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OUTER GRAVITATIONAL FIELD AND SHAPE OF THE PHYSICAL SURFACE OF THE EARTH

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A study is made of the conditions necessary for the solvability and uniqueness of solution of an integral equation by the aid of which the outer gravitational field and the shape of the physical surface of the earth may be determined.

A. Listing's Geoid and the Quasi-Geoid

An overwhelming majority of all geodetic surveys -- leveling, triangulation, astronomical and gravimetric determinations -- are made on the earth's surface. After much calculation, they are first reduced to a geoidal surface which is considerably more even. If these surveys embrace a large area, a second reduction is necessary from a geoid to an ellipsoid surface which is more convenient mathematically.

The system now adopted for reduction to a geoid and to an ellipsoid from a geoid is inadequate. It has possible inherent contradictions, since various parts of the terrestrial gravitational field are reduced on the basis of different hypotheses which are sometimes contradictory. Consequently, the resulting discrepancies can be of a systematic character; these discrepancies can noticeably reduce the accuracy of results in mountain regions or, after a geodetic translation, over mountain regions.

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In our opinion, the reduction problem in geodesy is mainly complicated by the traditional tendency to reduce observations to one level, i.e., to Listing's geoid coinciding with the average ocean level. Reduction to Brillouin's geoid, enveloping the whole terrestial mass but too far removed from a large part of the physical surface of the earth, is even more complicated. It must be admitted that reduction to Listing's geoid is a rational procedure; but up to the present it has been logical only so long as the results of gravimetric work were not applied in preparing geodetic observations and triangulation by the method of "rectifying" geodetic lines from a geoid to a ellipsoid, while retaining their lengths; so that a repetition of the projection of geodetic systems from a geoid to a ellipsoid would not hold true. But the possession of gravimetric surveys, already made in considerable detail over large areas of the whole earth, make essential corrections possible in the methods of studying the shape of the earth. It is also opportune to review the question of the expediency of retaining the Listing geoid as a basic surface to which or by means of which the geodetic elements measured are reduced to an ellepsoid.

With reference to this question the following basic circumstances must be considered:

1. The shape of Listing's geoid is generally not determinable, if density and disposition of the masses lying outside the geoid are not known. Therefore, the study of the shape of Listing's geoid is partly a geological task, since it cannot be solved, strictly speaking, before the completion of a geological investigation of all the continents.

2. Even with exhaustive geological data, a sufficiently accurate reduction to the geoid is connected with solving a complicated problem in the theory of potential because the reduction is carried out on an unknown surface of the earth, on which limited values are determined directly by observation.

3. By means of triangulation, now used in the USSR as a method of projection, the geoid is an intermediate reduction surface which must be excluded in the transition to a reference ellipsoid. Nevertheless, it leaves troublesome residual non-conformities because of the differences in reduction methods.

4. Geodesists are very rarely obliged to make observations of sufficient accuracy on the surface of the geoid. The location for an observation post with respect to a geoid is, strictly speaking, not known because it is only with a certain approximation that even orthometric heights can be considered as measured from the surface of Listing's geoid.

5. It is not necessary to connect the basic scientific problem of higher geodesy with the study of Listing's geoid. It would be more desirable to strive for the study of the outer gravitational field and the shape of the physical surface of the earth.

6. The shape of the physical surface of the earth can be determined with sufficient reliability on the basis only of data obtained from exact measurements, i.e., from the results of levelings and of measurements of gravity related to points with well-known approximate astronomic and geological or geophysical data need be involved in the principal solution of this problem. This circumstance does not exclude their usefulness, for example, in the interpolation of gravity and on many other occasions.

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7. Knowledge of the shape of the physical surface of the earth, under these conditions, supplies indispensable data for the solution of all practical problems of geodesy, particularly those originating in the high-precision measurements of degrees. Problems in reducing all geodetical elements measured to an ellipsoid can, on principal, be correctly solved.

The last two statements above will be corroborated later.

However, the use of the geoid undoubtedly had one positive side: it divided the most inaccurate part from the unitary, extremely complex, physical surface of the earth, a picture of which is given by certain levelings, i. e., height above sea level. There remained a second, incomparably smoother part, i. e., height of the geoid over the ellipsoid. Such a division is very natural and rational, and the geoid with both magnitudes divided has an additional simple physical meaning. But it is known that when such a division is achieved in practice a great number of obscurities and insurmountable difficulties arise. Furthermore, it will be shown that a similar division of the terrestial surface into irregular hypsometric and smooth geoidal parts can be made gradually, without recourse to Listing's geoid, by examining a certain surface close to the geoid. This surface, unlike the geoid, is determined on the basis of employing only data from exact geodetic measurements without depending on this or that notion of the structure of the earth.

As we shall see later, the surface in question is characterized by a disturbing potential on the terrestial surface, and its heights are obtained like the quotient by dividing the disturbing potential, at a given point of the earth's surface, by the normal value of gravity, calculated in a corresponding manner for this point. For the sake of definiteness we are obliged to introduce a new term for the surface in question; let us agree to call it a quasi-geoid. In the problem under consideration the quasi-geoid is introduced to separate the less smooth from the smooth parts of the earth. The former is determined by integration along the contour, and the second is obtained by solving a boundary problem in the theory of potential.

On the ocean plane, the quasi-geoid coincides with the geoid but on continents the quasi-geoid can be taken, if necessary, as an approximate expression of the geoid shape.

We must consider, first of all, how to separate the irregular part in the shape of the earth, which we shall call "the height of the point of the surface of the earth with reference to the quasi-geoid," or, more briefly, the "reference /vspomogatel'niy, literally auxiliary/ height." It would be advisable to determine the reference heights so that they would be sufficiently close to the orthometric heights. However, the usual orthometric correction does not entirely do away with the dependence of the result of leveling between two fixed points on the position of the guide line connecting them, which must have an effect on the dissimilarity in the heights when polygons of high-precision leveling are formed. The reference heights can easily be determined in a manner which will completely rid them of this defect.

B. Reference Heights

Let us consider the normal potential field U, formed by the "comparative earth" in which all masses are included inside the ellipsoidal surface of the level, characterized by the dimensions of the semiaxes a, a, b, the angular velocity of rotation ω and the value of gravity on its equator γ_e. The potential will now be uniquely determined at any outer point of space by the coordinates of this point. It is convenient to select for our purpose the coordinates described below. Let us draw through the specified point a coordinated line which will be a line of intersection passing through this point of the meridian plane and the hyperboloid focal to the leveled ellipsoid. The location of the point to be determined in space can then be described by an angle, tangent to this line, (drawn at the point of its intersection with the solid leveled ellipsoid) with the plane of the equator and the angle of the meridian plane, in which this tangent lies, with the plane of the original meridian, (latitude B*, longitude L*), and also length H* of the segment, of the line of force from the given point to the leveled ellipsoid.

We can regard the potential of the real earth

$$W(B^*, L^*, H^*) = W_0 - \int g\,dH$$

as known (correct to an additive constant W_0) at all points of the physical surface of the earth, and only on this surface; whereas it is not determinable at all other points of space without knowledge of the shape of the earth, and the density of the attracting masses, if it is a question of internal points.

Thus, we can also formulate an analytical expression for the disturbing potential T = W - U, but only for points of the physical surface of the earth. But since the shape of the earth is not known, the true coordinates of these points B*, L*, H* are unknown to us. However, we can consider the approximate value of the coordinates B, L, H as known, whereby the magnitudes

$$\left.\begin{aligned} \Delta B &= B^* - B \\ \Delta L &= L^* - L \\ \zeta &= H^* - H \end{aligned}\right\} \qquad (1)$$

are so small that their second powers and products may be disregarded. For this reason, in all further calculations, only terms of the first order relative to ΔB, ΔL, ζ are retained. With these assumptions, let us expand the expression for the normal potential into the Taylor series:

$$T(B^*, L^*, H^*) = W(B^*, L^*, H^*) - U(B^*, H^*) =$$
$$= W_0 - \int g\,dH - U(B, H) - \frac{\partial U(B,H)}{\partial H}\zeta - \frac{\partial U(B,H)}{\partial B}\Delta B\,. \qquad (2)$$

In this expression we can disregard the term

$$\frac{\partial U(3,H)}{\partial B}\Delta B \approx \frac{\partial^2 U(B,0)}{\partial H \partial B} H \Delta B,$$

because even when $\Delta B = 0'6$ an error less than 10^{-6} H will be introduced. It is advisable to determine the subsidiary height H with the help of the equation

$$-\int g\,dH = U(B,H) - U(B,0) = U(B,H) - U_0 \qquad (3)$$

which has a simple physical meaning. It indicates, namely, that in calculating the reference height H by the difference of the potentials, we assume that the potential field of the earth is normal. Under conditon (3) we obtain from (2) a relation analogous to Brown's well-known formula:

$$T(B^*, L^*, H^*) = -\frac{\partial U(B,H)}{\partial H}\zeta + W_0 - U_0 = \gamma(B,H)\zeta + W_0 - U_0 \qquad (4)$$

where γ(B,H) is the normal value of the acceleration of gravity at the point B, H, and the constant term $W_0 - U_0$ can, if desired, be reduced to zero by a proper selection of a surface of reference.

Equation (3), as far as its left-hand term can be considered as known, can be used in calculating H directly, if an exact analytical term is employed for the normal potential resulting from the theory of "Pitsetti" and "Somil'yan" [transliteration]. However, in order to facilitate calculations, it is better to expand it into a series which converges quickly in this case; besides, there are sufficiently detailed tables for these coefficients.

Thus, to determine H let us obtain the equation

$$-\int g\,dH = H\frac{\partial U}{\partial H} + \frac{1}{2}H^2\frac{\partial^2 U}{\partial H^2} + \frac{1}{6}H^3\frac{\partial^3 U}{\partial H^3} + \cdots$$

Granting, when H = 10 km, a relative error of not more than 1.10-6, in the terms of the second and third order, we need not consider compression and assume with above-mentioned accuracy that:

$$-\int g\,dH = -\frac{H}{1-\frac{H}{R}}\gamma(B,0), \tag{5}$$

where, as we know:

$$\gamma(B,0) = \gamma_e(1+\beta\sin^2 B - \beta_1\sin^2 2B),$$

and R is the average radius of the earth.

Starting from (5) it is not difficult to find a convenient corrective formula for our calculations for converting the difference of the observed heights of two points into the difference of their reference heights:

$$\delta(H_2-H_1) = \frac{1}{\gamma_m}\left[\sum \Delta g_0\Delta H + \sum(\gamma-\gamma_m)\Delta H + \frac{1}{2}(\gamma_1-\gamma_2)H_1+H_2)\right] - \frac{H_2^2-H_1^2}{R} \tag{6}$$

or

$$\delta(H_2-H_1) = \frac{1}{\gamma_m}\left[\sum \Delta g_B\Delta H + \sum(\gamma-\gamma_m)\Delta H + \frac{1}{2}(\gamma_1-\gamma_2)(H_1+H_2) + \frac{K}{2}(H_2^2-H_1)\right] - \frac{H_2^2-H_1^2}{R} \tag{7}$$

where Δg_0 is the anomaly in free air, calculated with reference to the normal formula $\gamma(B,0)$; Δg_B is the "Buge" [transliteration] anomaly, calculated with reference to the same normal formula with the coefficient K of dependence of the anomalies on the height; γ_2, γ_1 and $\gamma_m = \frac{\gamma_1+\gamma_2}{2}$ is the normal accelerationof gravity at the final, starting, and intermediate points.

The anomalies "in free air," or the Buge anomalies, serve in the last formula only for the interpolation of the values of gravity observed at discrete points at all intermediate points, located on the surface of the earth. Therefore, the difference between formula (6) and (7) consists only in a variation in the methods of interpolating anomalies, so that they can, with a very dense system of gravimetric points, produce identical results. However, in mountain districts, for a fixed density of points, formula (7) will lead to better results.

Hence, we have demonstrated that reference heights are completely determined only by the results of certain geodetic measurements and that they can be calculated more simply then orthometric heights. The difference of reference heights is a completely determined function of the initial and final leveling points and does not depend on the position of the guide line.

C. The Integral Equation Determining the Shape of a Quasi-Geoid

It is now necessary to show that the shape of a quasi-geoid can also be determined by the results of certain geodetical measurements, without recourse to any other data. In our determination of reference heights, the quasi-geoid heights are determined through the equation:

$$\zeta = \frac{T}{\gamma} + \frac{U_0-W_0}{\gamma} \tag{8}$$

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The value of the normal potential U_o on the ellipsoid of reference can be called known, if the semiaxes of this ellipsoid, the angular velocity of rotation and the normal value of gravity at one of its points, e. g., on the equator, are given.

The value of the potential W_o of the real earth at the initial point of levelings (the average ocean level) is unknown. Since the dimensions of the potential are expressed by the product of acceleration and length, it is obvious that it is impossible to determine W_o by means of pure geometry without having recourse to the results of linear measurements (the dimensions of the surface of reference cannot always be exactly fixed for the amount of centrifugal acceleration). Therefore, Stokes calculated the elevations of a geoid from the surface of reference, with a volume equal to the geoid, and Pitsetti calculated it from the surface of reference, for which $U_o = W_o$. In both cases the actual dimensions of the surface of reference remained unknown and depended on determination by the aid of degree measurements in which the linear measurements were the most important; it is easy to translate the values, calculated according to one method to values obtained by the other method.

N. R. Malkin (1) finds a nonconformity with Stokes' solution when the term $U_o = W_o$ in "Bruns" formula is present. It is possible to agree, but only according to the interpretation given above, connecting this case with the obvious fact of the impossibility of determining the dimensions of the earth by gravimetric observations alone. The additional term depending on $U_o - W_o$ introduced in the Stokes' formula by Malkin has a simple meaning: it expresses the change in an elevated geoid in transition from the surface reading in which the potential equals U_o to a surface in which the potential equals W_o. Increasing from this point, the additional effect on the deflection of a plumb line expresses the distortion of the line of force of the normal field on a segment from one surface reading to the other.

The normal value of gravity γ at the magnitude assigned to it at the equator and for a given oblateness does not depend in practice on an error in the major semiaxis of a normal ellipsoid. Hence, without changing γ and leaving the reference heights unchanged, we may assume that a has a value such that in (8) $U_o = W_c$.

Thereby we express the fact that it is practically impossible to determine W_o or a without bringing in the results of linear measurements and that ζ is to be read from an ellipsoid with an unknown major semiaxis on the surface of which the normal potential equals W_o. After defining the elevations of the quasi-geoid with respect to this ellipsoid, we may pass at will to another ellipsoid such, for example, that its volume is equal to the volume of a quasi-geoid.

Accordingly, we shall assume for the present that $U_c = W_o$; it is possible to get rid of this assumption without any special difficulty and this will be done later.

The search for ζ, consequently, amounts to determining T on the physical surface of the earth, because now:

$$\zeta = \frac{T}{\gamma} . \tag{9}$$

From the definition of the disturbing potential it follows:

$$\frac{\partial T}{\partial \nu} = \frac{\partial W}{\partial \nu} - \frac{\partial U}{\partial \nu} .$$

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Here derivatives are taken with respect to the direction of the coordinate line ν, with a relative error of the order of the square of deflection from the perpendicular, the value $-\frac{\partial W}{\partial \nu}$ on the earth's surface coincides with the observed value of gravity g and consequently:

$$\left(\frac{\partial T}{\partial \nu} - \gamma\right)_{H+\zeta} = -g$$

With the same degree of accuracy, we obtain:

$$\left(\frac{\partial T}{\partial \nu}\right)_{H+\zeta} = \left(\frac{\partial T}{\partial \nu}\right)_H, \quad (\gamma)_{H+\zeta} = (\gamma)_H + \zeta\frac{\partial \gamma}{\partial \nu},$$

so that

$$-(g-\gamma) = \left(\frac{\partial T}{\partial \nu} - \frac{T}{\gamma}\cdot\frac{\partial \gamma}{\partial \nu}\right)_H. \tag{10}$$

Now finding T is reduced to solving the third boundary problem of the theory of potential: the function of T must now be such that it satisfies conditions (10) on surface S and it must be a harmonic function of the coordinates outside S and be regular at infinity. In this problem, surface S is the first approximation to the earth's shape resulting from the reading of the reference heights only above the ellipsoid of reference.

In short, the solution of this problem is like that of (2). Assuming that the specified point is located on surface S, we apply Green's formula to the disturbing potential:

$$T = -\frac{1}{2\pi}\int_S\left(\frac{\partial T}{\partial n}\cdot\frac{1}{r} - T\,\frac{\partial \frac{1}{r}}{\partial n}\right)dS.$$

We shall express the derivative $\frac{\partial}{\partial n}$ with respect to the direction of the normal to S by the derivative with respect to the direction of the coordinate line $\frac{\partial}{\partial \nu}$ and the derivative with respect to the direction τ of that tangent to S which lies in the same plane as the directions n and ν:

$$\frac{\partial}{\partial n} = \sec\alpha\frac{\partial}{\partial \nu} - tg\,\alpha\frac{\partial}{\partial \tau},$$

where α is the angle between the directions n and ν.

In orthogonal curvilinear coordinates H, B, L in which the differential of the linear element dl is expressed by the quadratic form:

$$dl^2 = h^2 dH^2 + h_1^2\,dB^2 + h_2^2\,dL^2,$$

H is connected with B and L by the equation of surface S and the derivative with respect to τ is defined by formula:

$$tg\,\alpha\frac{\partial F}{\cos\alpha\,\partial\tau} = \frac{h}{h_1^2}\cdot\frac{\partial F}{\partial B}\cdot\frac{\partial H}{\partial B} + \frac{h}{h_2^2}\cdot\frac{\partial F}{\partial L}\cdot\frac{\partial H}{\partial L}.$$

Here, in the differentiation of B and L, the function F is regarded as a function of only two of these arguments (inasmuch as the dependence of H on B and L is determined by the form surface S).

Introducing for the sake of brevity the operator $\bar{D}$ (F, H), which symbolizes operation upon F and H effected in the right-hand side of the last equation, we shall obtain:

$$tg\,\alpha\frac{\partial F}{\partial \tau} = \bar{D}(F,H)\cos\alpha.$$

so that

$$\frac{\partial F}{\partial n} = \sec\alpha \frac{\partial F}{\partial \nu} - \bar{D}(F, H) \cos\alpha . \tag{11}$$

If we introduce a second differential operator

$$\Delta_2 F = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial B}\left(\frac{h h_2}{h_1} \cdot \frac{\partial F}{\partial B}\right) + \frac{\partial}{\partial L}\left(\frac{h h_1}{h_2} \cdot \frac{\partial F}{\partial L}\right)\right]$$

it will be possible to prove the absolute identity of:

$$\int_S \bar{D}(F, H) \cos\alpha \, dS = -\int_S F \Delta_2 H \cos\alpha \, dS. \tag{12}$$

It is now possible in Green's formula to express $\frac{\partial T}{\partial \tau}$ by T and thereby to obtain a linear integral equation for T.

Making use of boundary condition (10), we reduce the problem under consideration by the above method to the following integral equation of the Fredholm type of the second order with a peculiar unsymmetrical integrand:

$$T = \frac{1}{2\pi}\int \frac{q-\Upsilon}{r} \sec\alpha \, dS + \frac{1}{2\pi}\int_S T\left[\frac{\partial \frac{1}{r}}{\partial \nu} \sec\alpha - \frac{1}{r\gamma} \cdot \frac{\partial \gamma}{\partial \nu} \sec\alpha - \right.$$

$$\left. - 2\bar{D}\left(\frac{1}{r}, H\right)\cos\alpha - \frac{1}{r}\Delta_2 H \cos\alpha\right] dS . \tag{13}$$

D. Conditions Governing the Existence of a Solution

Study of equation (13) is complicated by the fact that its integrand becomes infinite for r = 0 and as part of an integral equation it exists only in the sense of Cauchy's principle value. Hence, it is not possible always to transpose the order of two successive integrations; chiefly for this reason, the conclusions of the general theory must be accepted only with great caution. Inasmuch as the singular point of the integrand is a pole of the second, it is impossible to affirm that, after the last repetition, the repeated integrand will be finite.

With the aid of (11) it is possible to write this equation in such a form as:

$$2\pi T \int_S T\left[\frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r\gamma} \cdot \frac{\partial \gamma}{\partial \nu} \sec\alpha - \bar{D}\left(\frac{1}{r}, H\right)\cos\alpha - \right.$$

$$\left. - \frac{1}{r}\Delta_2 H \cos\alpha\right] dS = -\int \frac{q-\Upsilon}{r} \sec\alpha \, dS. \tag{14}$$

Let us construct a homolous integral equation, combined with that given, of such a nature that integration in it will be carried out with respect to the second argument, with respect to the coordinates of the specified point; and that the first argument will be regarded as a fixed parameter with respect to integration over the surface.

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Let us remember that $\frac{1}{r}$ in this equation is a function of two arguments -- a function of the coordinates of the present point to which the element of surface dS is related and a function of the location of the specified point, the coordinates of which enter into $\frac{1}{r}$ as parameters. The expressions α, H, γ, $\frac{\partial \gamma}{\partial \nu}$, and the operators $\frac{\partial}{\partial n}$ and $\bar{D}$ as well, of course, depend on the coordinates of the element of integration and not on the second argument.

Taking these remarks into consideration and denoting by μ the unknown function of the combined equation, we obtain:

$$2\pi\mu = \int_S \mu\left(\frac{\partial \frac{1}{r}}{\partial n_1}\right) dS - \left(\frac{1}{\gamma}\cdot\frac{\partial \gamma}{\partial \nu}\sec\alpha + \cos\alpha\, \Delta_2 H\right)\int_S \frac{\mu}{r}\, dS - \\ - \cos\alpha \int_S \mu \bar{D}\left(\frac{1}{r}; H\right) dS.$$

Now $\frac{\partial}{\partial n_1}$ denotes differentiation with respect to the direction of the normal to the surface of the earth at a specified point and the operator $\bar{D}_0$ is a symbol of differentiation with respect to the second argument (being the parameter in the combined equation) in the constant plane tangent to the surface of the earth at a specified point.

Changing the order of differentiation and integration of the parameters in the last equation, we can write it in a different form:

$$2\pi\mu = \frac{\partial}{\partial n_1}\int \frac{\mu}{r} dS - \left(\frac{1}{\gamma}\cdot\frac{\partial \gamma}{\partial \nu}\sec\alpha + \cos\alpha\,\Delta_2 H\right)\int\frac{\mu}{r}\, dS - \\ - \cos\alpha\, \bar{D}\left(\int\frac{\mu}{r}\, dS, H\right). \tag{15}$$

For the solution of equation (14), it is necessary that any solution of equation (15) should be orthogonal to the free term of equation (14), satisfying the condition:

$$\int \mu\left[\frac{g-\gamma}{r}\sec\alpha\, dS\right] dS = 0. \tag{16}$$

In fact, multiplying equation (14) by μ and integrating over the whole surface S, we obtain:

$$2\pi\int\left[T\mu - \mu\int T\left[\frac{\partial\frac{1}{r}}{\partial n} - \frac{1}{r\gamma}\cdot\frac{\partial\gamma}{\partial\nu}\sec\alpha - \right.\right. \\ \left.\left. - \frac{\cos\alpha}{r}\Delta_2 H - \bar{D}\left(\frac{1}{r}, H\right)\cos\alpha\right] dS\right] dS = \int\mu\left[\int\frac{g-\gamma}{r}\sec\alpha\, dS\right] dS.$$

Changing the order of integration in the left-hand side, which is possible because one of the integrals is an ordinary one, and integrating the first with respect to the second argument, we find that if μ satisfies equation (15), the left side will be reduced to zero and, consequently, condition (16) should hold true. It is for this reason that the necessity for this condition arises.

Proving the adequacy of condition (16) in solving equation (14) is much more complicated.

V. D. Kupradze's article on "Some Peculiar Equations of Mathematical Physics" [3] quoted the basic results of G. Giraud's research. Giraud proved for equations of a similar type the following theorems, which are well-known from the theory of regular equations:

(1) A finite number of linearly independent solutions of a homologous equation corresponds to any pole of a resolvent;

(2) A combined equation has the same number of linearly independent solutions;

(3) The necessary and sufficient condition for solving a homologous equation is the orthogonal character of the right-hand side of the equation with respect to all solutions of a homologous combined equation.

It follows from the last theorem that fulfilment of condition (16) is both necessary and sufficient.

Changing the order of integration in (16), we can put it in the following form:

$$\int \lambda (g-\gamma) \sec \alpha \, dS = 0, \tag{17}$$

where

$$\lambda = \int \frac{\mu}{r} \, dS. \tag{18}$$

Now equation (15) is written as follows:

$$\frac{\partial \lambda}{\partial n_0} - 2\pi\mu = \left(\frac{1}{\gamma} \cdot \frac{\partial \gamma}{\partial \nu} \sec \alpha + \cos \alpha \, \Delta_2 H\right) \lambda - \cos \alpha \, \bar{D}(\lambda, H). \tag{19}$$

Let us try to omit μ from this equation since in condition (17) only λ enters. For this purpose, let us bear in mind that λ may be interpreted as the potential on the surface S of a simple layer with density μ, as shown by equation (18). Derivatives of the potential of a simple layer with respect to the direction of the tangents to the surface are continuous. The normal derivative of the potential of a simple layer on surface S experiences a discontinuity; whereupon the discontinuity and the value of $\frac{\partial \lambda}{\partial n_0}$ on the surface are determined by the well-known formula of Poisson and Fimell [transliteration]:

$$\frac{\partial \lambda}{\partial n_i} - \frac{\partial \lambda}{\partial n_e} = 4\pi\mu,$$

$$\frac{1}{2}\left[\frac{\partial \lambda}{\partial n_i} + \frac{\partial \lambda}{\partial n_e}\right] = \frac{\partial \lambda}{\partial n_0}.$$

Hence the left-hand side of equation (19) equals $\frac{\partial \lambda}{\partial n_e}$.

The value of λ and $\frac{\partial \lambda}{\partial n_e}$ on surface S are connected with Green's formula:

$$\lambda = -\frac{1}{2\pi} \int \left[\frac{1}{r} \cdot \frac{\partial \lambda}{\partial n_e} - \lambda \frac{\partial \frac{1}{r}}{\partial n}\right] dS.$$

Substituting the right side of (19) for $\frac{\partial\lambda}{\partial n_e}$ here, we obtain a linear integral equation which must be satisfied by the function of λ:

$$\lambda = -\frac{1}{2\pi}\int\left\{\lambda\left[\frac{1}{r\gamma}\cdot\frac{\partial\gamma}{\partial\nu}\sec\alpha + \frac{\cos\alpha}{r}\Delta_2 H - \frac{\partial\frac{1}{r}}{\partial n}\right] - \frac{\cos\alpha}{r}\bar{D}(\lambda, H)\right\} dS.$$

It is easy to simplify this equation. In fact through (12):

$$\int\frac{\lambda}{r}\cos\alpha\,\Delta_2 H\,dS = -\int\bar{D}\left(\frac{\lambda}{r}, H\right)\cos\alpha\,dS.$$

It follows from the definition of the operator $\bar{D}$ that:

$$\bar{D}\left(\frac{\lambda}{r}, H\right) = \lambda\,\bar{D}\left(\frac{1}{r}, H\right) + \frac{1}{r}\bar{D}(\lambda, H).$$

Taking this into consideration and substituting for $\frac{\partial\frac{1}{r}}{\partial n}$ its value as obtained according to (11), we shall find that:

$$2\pi\lambda = \int\lambda\left(\frac{\partial\frac{1}{r}}{\partial\nu} - \frac{1}{r\gamma}\cdot\frac{\partial\gamma}{\partial\nu}\right)\sec\alpha\,dS. \qquad (20)$$

is a linear integral equation with respect to λ (again with a special integral). We are now more interested in the function of μ. In fact, for each solution of (15) there is a single value of the function of λ, since assignment of the surface simply determines the potential of a simple layer. On the other hand, search for μ according to assigned values of λ, that is, the density of the surface layer according to the value of its potential on the surface amounts to the solution of Dirichlet's outer and inner problem, since the density of the layer can be expressed by the difference between the outer and inner derivatives and the potential of this layer. As we know, this problem is always capable of solution and, moreover, of a unique solution when the hypotheses about the properties of surface S are sufficiently broad. Thus, λ and μ are simply interconnected and the number of linearly independent solutions (15) and (20) are identical. The latter circumstance has an essential significance for us. Inasmuch as the number of linearly independent solutions of a homologous equation corresponding to the complete equation (14) and the number of such solutions of equation (15) combined with it are identical, it can be affirmed that the number of linearly independent solutions of homologous equation (20) is equal to the number of linearly independent solutions of the homologous equation obtained from (14) by eliminating the free term. Since the question of solutions for a corresponding homologous equation has an essential significance in studying the conditions for solving (14), we shall explain the number of such linearly independent solutions of (20). For this purpose, let us set up an equation combined with (20). Proceeding analogously, as was done earlier, and calling the unknown function of the combined equation φ, we shall have:

$$2\pi\varphi = \sec\alpha\frac{\partial}{\partial\nu}\int\frac{\varphi}{r}\,dS - \sec\alpha\cdot\frac{1}{\gamma}\cdot\frac{\partial\gamma}{\partial\nu}\int\frac{\varphi}{r}\,dS \qquad (21)$$

Considering φ as the surface density of a simple layer and introducing the function ψ, the potential of this layer at any point of surface S, we obtain:

$$2\pi\varphi = \sec\alpha\frac{\partial\psi}{\partial\nu} - \frac{\sec\alpha}{\gamma}\cdot\frac{\partial\gamma}{\partial\nu}\psi. \qquad (22)$$

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Let us bear in mind that the discontinuity of the derivative of the potential of the simple layer with respect to the direction forming the angle α with the normal, is equal to $4\pi\varphi\cos\alpha$. Hence

$$\frac{\partial\psi}{\partial\nu} - 2\pi\varphi\cos\alpha = \frac{\partial\psi}{\partial\nu_e}. \tag{23}$$

Inasmuch as $\cos\alpha \neq 0$, instead of (22) we shall obtain:

$$\frac{\partial\psi}{\partial\nu_e} = \frac{1}{\gamma}\cdot\frac{\partial\gamma}{\partial\nu_e}\psi. \tag{24}$$

This equation is equivalent to (21) in the sense that for each solution of one of these equations there is a corresponding unique solution of another equation which, as before, follows from the uniqueness of the solution of Direchlet's problem.

From (24) we find that:

$$\frac{\partial\ln\psi}{\partial\nu_e} = \frac{\partial\ln\gamma}{\partial\nu_e} \tag{25}$$

or

$$\psi = \Phi(B^*, L^*)\,\gamma. \tag{26}$$

The solution $\psi = 0$ does not interest us.

Thus, (21) is capable of a solution differing from zero if there are harmonic functions satisfying condition (25). For each of such harmonic functions there is a corresponding solution, differing from zero, for the solution of the homologous equation obtained from (14). It is easy to see that this condition is sufficient directly from the original equation (14). Once the harmonic function satisfies (25), it also satisfies (24). This function will also satisfy (14) if the anomalies are correspondingly expressed.

In fact, since

$$g - \gamma = -\frac{\partial T}{\partial\nu} + \frac{1}{\gamma}\cdot\frac{\partial\gamma}{\partial\nu}T, \tag{10}$$

we may feel sure that when $T = \psi$, on the basis of (24), $g - \gamma = 0$ and consequently ψ is the solution of a homologous equation corresponding to the left side of (14).

In the general case where the surface of reference is of a complicated structure, the problem of the existence of a harmonic solution satisfying condition (25) has not been completely studied by us. The simplest case has the greatest practical value, where the derivative $\frac{\partial\gamma}{\partial\nu}$ is considered constant (the coefficient of reduction in free air). In a pure form it corresponds to the hypothesis that the surface of reference is a sphere.

Thus, if the surface of reference is a sphere, $\gamma = \frac{M}{\rho^2}$, where ρ is the distance from the origin of the coordinates and M is a constant proportional to the mass of the earth. Since the potential ψ outside S should satisfy Laplace's equation, we should in this case, consider that

$$\Phi(B^*, L^*) = \frac{Y_1}{M} \text{ and } \psi = \frac{Y_1}{\rho^2}.$$

where Y_1 is any spherical function of the first order. This means that for ψ we have three linearly independent solutions, since an arbitrary spherical function of the first order is the sum of three linearly independent functions.

Hence, in this case equation (24) and, consequently, (21), (20), (10), and, finally, the homologous equation obtained from (14) have three linearly independent solutions apiece.

E. Physical Meaning of Solvability of Conditions

Thus, when the surface of reference is a sphere, there are three linearly independent functions λ_1, λ_2 and λ_3, which satisfy equation (20). Assigning to the surface the combined values $g-\gamma$ for the solvability of (14) should satisfy three conditions of the type (17). The value $g-\gamma$ has, in fact, been obtained from observations, measurements of gravity, leveling, and astronomical work. Will the conditions of (17) be fulfilled if the measurements are made with perfect accuracy?

To answer this question let us carry the study of (20) further. Turning to spherical coordinates let us introduce instead of λ a new function σ, connected λ by the relation:

$$\lambda \sec \alpha = \rho \sigma . \tag{27}$$

We shall have as a result:

$$\frac{1}{\gamma} \frac{\partial \gamma}{\partial \nu} = -\frac{2}{\rho} .$$

Now equation (20) can be put in the form:

$$2\pi\sigma \cos \alpha = \frac{1}{\rho_0} \int \sigma \left(\frac{2}{r} + \rho \frac{\partial \frac{1}{r}}{\partial \rho} \right) dS . \tag{28}$$

Denoting by ρ_0 the value of the radius vector for a specified point and by ψ the angle between ρ_0 and ρ, let us make use of the well-known relationship between these magnitudes and ρ :

$$r^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos \psi .$$

Hence:

$$\left. \begin{aligned} \frac{\partial \frac{1}{r}}{\partial \rho} = \frac{\rho_0 \cos \psi - \rho}{r^3}, \quad \frac{\partial \frac{1}{r}}{\partial \rho_0} = \frac{\rho \cos \psi - \rho_0}{r^3}, \\ \frac{1}{r} + \rho \frac{\partial \frac{1}{r}}{\partial \rho} = -\rho_0 \frac{\partial \frac{1}{r}}{\partial \rho_0} . \end{aligned} \right\} \tag{29}$$

Now, in place of (28) we obtain:

$$2\pi\sigma \cos \alpha = \frac{1}{\rho_0} \int \frac{\sigma}{r} dS - \frac{\partial}{\partial \rho} \int \frac{\sigma}{r} dS . \tag{30}$$

The term $2\pi\sigma\cos\alpha$ on the left-hand side equals half the gap in the discontinuity of the derivative of the potential of a simple layer of density σ with respect to the direction forming the angle α with the normal, that is, the direction ρ_0. Therefore, introducing the auxiliary function U_1, the potential of a simple layer of density σ at a point lying outside S, let us put (30) in the form:

$$U_1 = \int \frac{\sigma}{r} dS \tag{31}$$

where

$$\frac{\partial U_i}{\partial \rho} = \frac{U_i}{\rho}.$$

The outer derivative is taken off the left-hand side of the equation. The function

$$U_i = Y_1 \rho \tag{32}$$

outside the surface satisfies Laplace's solution and equation (31). The Y of Laplace, Y_1, depends on three linearly independent functions, to each of which there corresponds a unique and completely definite value of the density σ.

In fact,

$$\sigma = \frac{1}{4\pi}\left(\frac{\partial U_i}{\partial n} - \frac{\partial U_e}{\partial n}\right), \tag{33}$$

where U_e is the solution of Direchlet's outer problem corresponding to the boundary value of U_i. Since a single combination of values $\frac{\partial U_e}{\partial n}$ corresponds to the assigned combination of values for U_1 on surface S, a unique and fully definite value of σ will correspond to each value of U_1. It is possible to reach this result, if we study the equation obtained from (30) and (32):

$$2\pi\sigma\cos\alpha = Y_1 - \frac{\partial}{\partial\rho}\int\frac{\sigma}{r} dS. \tag{34}$$

As a result of the transition from λ to σ, with the aid of (27) let us put the condition for solvability in the following form:

$$\int \sigma (g - \gamma)\rho dS = 0, \tag{35}$$

or, eliminating σ by the aid of (33), in the form:

$$\int\left(\frac{\partial U_i}{\partial n} - \frac{\partial U_e}{\partial n}\right)(g-\gamma)\rho dS = 0. \tag{36}$$

Let us study the function

$$V = \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}(\rho^2 T). \tag{37}$$

On surface S and by virtue of (10)

$$V = -(g - \gamma)\rho; \tag{38}$$

outside the surface S

$$\Delta V = 0. \tag{39}$$

If we assume the existence of three derivatives of the function T, that is, that the volume density of the masses generating potential T is a differentiable continous function of the coordinates, it can be shown that for the volume of a finite surface S

$$\int \rho Y_1 \Delta V d\tau = 0. \tag{40}$$

Now, making use of the equation

$$\rho \Delta \left(\rho \frac{\partial T}{\partial \rho} \right) = \frac{\partial}{\partial \rho} (\rho^2 \Delta T),$$

readily verified by the aid of Laplace's well-known expression for the operator in spherical coordinates, we obtain

$$\Delta V = 4 \Delta T + \rho \frac{\partial}{\partial \rho} \Delta T. \tag{41}$$

Substituting (41) in (40) and considering that: $d\tau = \rho^2 d\rho\, d\omega$,

where $d\omega$ is an element of a solid angle, we find that:

$$\int \rho Y_1 \Delta V d\tau = 4 \iint \rho^3 Y_1 \Delta T d\rho d\omega + \iint \rho^4 Y_1 \frac{\partial}{\partial \rho} \Delta T d\rho d\omega.$$

Let us convert the second integral on the right by integration by parts:

$$\int_0^\rho \rho^4 Y_1 \frac{\partial}{\partial \rho} \Delta T d\rho = \left| \rho^4 Y_1 \Delta T \right|_0^\rho - 4 \int_0^\rho \rho^3 Y_1 \Delta T d\rho;$$

we shall then have:

$$\int \rho Y_1 \Delta V d\tau = \int_S \rho^4 Y_1 \Delta T d\omega.$$

On the basis of the hypothesis of continuity of ΔT, we must assume that $\Delta T = 0$ also on surface S and, therefore, that the right side of the last equation has been shown to be equal to zero; this proves the correctness of the assertion made in formula (40).

Let us study condition (36), writing it with the aid of (38) in the following manner:

$$\int \left(\frac{\partial U_i}{\partial n} - \frac{\partial U_e}{\partial n} \right) V dS = 0.$$

The functions V and U_e, harmonic outside S, are regular at infinity, and therefore:

$$\int \left(V \frac{\partial U_e}{\partial n} - U_e \frac{\partial V}{\partial n} \right) dS = 0.$$

Adding the last two equations and bearing in mind that $U_i = U_e$ on the surface S, we eliminate the unknown function $\frac{\partial U_e}{\partial n}$ and obtain:

$$\int \left(V \frac{\partial U_i}{\partial n} - U_i \frac{\partial V}{\partial n} \right) dS = 0.$$

Converting this surface integral by means of Green's formula to a volume integral and noting the harmonic properties of the function U_1 inside S, let us reduce the condition under consideration to the form:

$$\int U_i \Delta V d\tau = 0. \tag{42}$$

But we have shown that $U_1 = Y_1 \rho$, and therefore condition (42) coincides with the demonstrated equation (40). Just as three arbitrary parameters form part of Y_1, so all three conditions as we have seen, superimposed on the anomolies $g-\gamma$, are themselves satisfied.

Thus, if the boundary values $g-\gamma$ are not arbitrarily assigned and have a definite physical significance, being obtained by observations and, consequently, corresponding to some distribution of masses, the conditions for the existence of a solution of our problem must be satisfied; these conditions can govern the accuracy of observations from which the boundary values are obtained.

The hypothesis of the differentiability of volume density can possibly be eliminated by further extensive demonstrations. But in this physical problem this hypothesis seems entirely permissible to us. In fact, the number of surfaces of discontinuity in the density inside the earth is finite and, consequently, in a sufficiently fine layer, it is possible to redistribute the masses in such a way that the density at any point can be differentiated. At the same time, the change of the outer potential field, as a result of the redistribution of masses, will be less than any previously assigned magnitude if a sufficiently small thickness of the layer in which the redistribution of masses takes place, is selected.

F. <u>Conditions for a Unique Solution</u>

Returning to (14), we can now verify the fact that when the boundary values are correctly obtained this equation can always be solved but that its general solution comprises three arbitrary constants in accordance with which the solution of a homologous equation is made up of three linearly independent functions. It is not difficult to obtain this general solution.

Let T_1 be the particular solution of (14) and T_2 be the general solution of the homologous equation corresponding to it. Then general solution of (14) will be equal to the sum:

$$T = T_1 + T_2.$$

The reason for the second term is obvious. In fact, let us displace the sphere of reference in some direction; it is evident that for this displacement we are providing for three degrees of freedom. Now the potential developed by the earth of comparison at any point of space in S and outside S changes to the magnitude $\frac{Y_1}{\rho^2}$, where Y_1 is Laplace's Y of the first order and dependent upon three constants: three coordinates of the center of the displace surface of reference relative to the original position of its center. A new value of the disturbing potential and a new value for $\frac{\partial T}{\partial \rho}$ will correspond to the new position of the surface of reference.

But thereby the sum

$$\frac{\partial T}{\partial \rho} + \frac{2T}{\rho} = -(g-\gamma)$$

is not changed. Consequently, for the new position of the surface of reference:

$$2\pi\left(T_1+\frac{Y_1}{\rho^2}\right)=\int\frac{g-\gamma}{r}\sec\alpha\,dS+\int\left(T_1+\frac{Y_1}{\rho^2}\right)\left(\frac{\partial\frac{1}{r}}{\partial n}+\dots\right)dS.$$

After calculating therefrom equation (14), which by agreement is satisfied by the function T_1, we may feel certain that the function $\frac{Y_1}{\rho^2}$ satisfies the corresponding homologous equation. Therefore,

$$T_2=\frac{Y_1}{\rho^2}$$

and the general solution of (14) is such that:

$$T=T_1+\frac{Y_1}{\rho^2}. \tag{43}$$

The correctness of the result obtained is easily verified by direct calculation, but we shall not do this

Eliminating the condition $U_0 = W_0$ in the last equation giving the general solution of the initial equation (14), we obtain the general expression for ζ :

$$\zeta=\frac{T_1}{\gamma}+Z_1+\frac{U_0-W_0}{\gamma}, \tag{44}$$

where T_1 is the particular solution of (14) and Z_1 is a derivative, spherical function of the first order.

The physical significance of the multiplicity of solutions obtained is clear: here, just as in the general case, the dimensions and position of the surface of reference remain indeterminate. Consequently, in this case also it is possible to introduce a condition that the volumes of the quasi-geoid and the surface of reference be equal. Such a condition is geometrically more obvious than the equality of potentials U_0 and W_0 or the equality of the masses. Moreover, it is possible to demand the combination of mass centers or, again in the interest of geometrical obviousness, to combine the centers of the volumes of the quasi-geoid and the surface of reference. It is easy to express these geometrical conditions in analytical form.

The condition of volume equality

$$\int\zeta\,d\sigma=\int\frac{T_0+(U_0-W_0)}{\gamma}\,d\sigma=0$$

permits determining $U_0 - W_0$ (the term with Z_1 drops out). Here $d\sigma$ is an element of the sphere's surface.

The condition for combining th volume centers

$$\int\zeta\cos\psi\,d\sigma=\int\left(\frac{T_1}{\gamma}+Z_1\right)\cos\psi\,d\sigma+(U_0-W_0)\int\frac{\cos\psi}{\gamma}\,d\sigma$$

[redacted]

where ψ is calculated from an arbitrary point on the sphere, permits defining all three coefficients entering into Z_1. The magnitude of the second integral on the right side of the equation is negligible, and hence the position of the center of the volume can be determined almost independently of $U_0 - W_0$.

However, when the volume centers are combined, the rotation axes of the real earth and the earth of comparison do not coincide and the potential of centrifugal force enters, to a slight degree, into the disturbing potential. Hence it is better to demand the coincidence of the center of inertia of the earth with the center of the sphere of reference.

This condition is expressed in the following form:

$$\int \Delta T \rho Z_1 \, d\tau = 0,$$

where Z_1 is any spherical function of the first order. By means of Green's formula the volume integral on the left-hand can be converted into a surface integral:

$$\int\left[\frac{\partial T}{\partial n}\rho Z_1 - T\frac{\partial}{\partial n}(\rho Z_1)\right] dS = 0.$$

If we substitute here T from (43) and include as before $\frac{\partial T}{\partial n}$ and utilize the fact that the three coefficients entering into Z_1 are arbitrary, we obtain three equations for the determination of the three arbitrary constants entering into (43).

G. Representing Earth's Shape by Density of the Surface Layer

Let us turn to equation (21) and examine it and the heterogeneous equation

$$2\pi\varphi \cos\alpha - \frac{\partial}{\partial \nu}\int \frac{\varphi}{r} dS + \frac{1}{\gamma}\cdot\frac{\partial \gamma}{\partial \nu}\int \frac{\varphi}{r} dS = g - \gamma. \qquad (45)$$

It is evident that (20) will combine with it; but the condition of solvability, formula (17), is general for this equation and for the original equation (13).

Consequently all conclusions reached in studying (13) are fully applicable to (45).

Let us introduce the auxiliary function T' -- the potential of a simple layer located on surface S of density φ :

$$T' = \int \frac{\varphi}{r} dS. \qquad (46)$$

The value on surface S derived from T' in the direction m, forming the angle (n, m) with the direction n of the outer normal to S, is expressed by the well-known formula

$$\frac{\partial T'}{\partial m} = \frac{\partial}{\partial m}\int \frac{\varphi}{r} dS - 2\pi\varphi \cos(n, m). \qquad (47)$$

With the aid of (46) and (47), equation (45) can be written in the form:

$$\frac{\partial T'}{\partial \nu_e} - \frac{1}{\gamma} \cdot \frac{\partial \gamma}{\partial \nu} T' = -(g-\gamma). \tag{48}$$

Calculating (48) from (10) and following the reasoning given at the end of Section D, we may feel certain that, in a case like that studied by use of a spherical surface of reference:

$$T - T' = \frac{\gamma_1}{\rho^2}.$$

Comparing this result with (43), we reach the conclusion that any solution of (45), after substitution in (46), leads to the particular solution of equation (13).

Hence the two equations (45) and (46) are equivalent to equation (13).

For the two surfaces of reference we obtain by the aid of (29):

$$\left.\begin{aligned} \frac{1}{\gamma} \cdot \frac{\partial \gamma}{\partial \nu} &= -\frac{2}{\rho}, \\ \frac{\partial \frac{1}{r}}{\partial \rho_0} + \frac{1}{2\rho_0 r} &= \frac{\rho^2 - \rho_0^2}{2\rho_0 r^3}. \end{aligned}\right\} \tag{49}$$

Now equation (45) is greatly simplified:

$$2\pi\varphi\cos\alpha = (g-\gamma) + \frac{3}{2\rho_0}\int \frac{\varphi}{r}\, dS + \frac{1}{2\rho_0}\int \frac{\rho^2 - \rho_0^2}{r^3}\varphi\, dS. \tag{50}$$

After φ has been found from the solution of this equation, the height of the quasi-geoid may be obtained from (46); and (47) will permit determination of deflections of a plumb line on the physical surface of the earth.

After determining the heights of the quasi-geoid and the deflection of the plumb line, we can pass from surface S representing the shape of the earth in the first approximation, to the new surface S', obtained by adding the heights of the quasi-geoid to the reference heights and also by correcting the astronomical coordinates by means of the magnitudes obtained for deflection of a plumb line. The new surface S' will characterize the shape of the earth in a second approximation; if necessary, further approximations can be obtained.

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