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Letter to the editor

On the validity of "A proof that the discrete singular convolution (DSC)/Lagrange-distributed approximation function (LDAF) method is inferior to high order finite differences"

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Recently, Boyd published a paper entitled "A proof that the discrete singular convolution (DSC)/ Lagrange-distributed approximating function (LDAF) method is inferior to high order finite differences" (J. Comput. Phys. 214 (2006) 538–549 [3]), which will be referred to as "Proof". This Letter analyzes the validity of central claims in "Proof".

In its title and abstract, "Proof" refers to LDAF as "Lagrange-distributed approximating function", while in its introduction, "Proof" refers to LDAF as "linear distributed approximating functional" and attributes it to Hoffman et al. [4]. In fact, "linear distributed approximating functional (LDAF)" does not exist. The paper by Hoffman et al. [4] was concerned the "distributed approximating functional". "Proof" did not give any detailed analysis, theoretical expression, and correct literature reference about LDAF. Moreover, the DSC algorithm can be realized by many different kernels that may behave very differently from each other [8,9]. What is discussed in "Proof" is a special case, the regularized Shannon kernel (RSK) $\delta(x-x_j)=\frac{\sin\frac{\pi}{h}(x-x_j)}{\frac{\pi}{h}(x-x_j)}\exp(-a^2[x-x_j]^2/h^2)$, where h is the grid spacing. We therefore limit our discussion to the DSC-RSK method.

First, "Proof" states in its abstract that "we show that the DSC is worse than the standard finite differences in differentiating $\exp^{(ikx)}$ for all k when $a \geqslant a_{\mathrm{FD}}$ where $a_{\mathrm{FD}}=1/\sqrt{N+1}$ with N as the stencil width is the value of the DSC parameter that makes its weights most closely resemble those of finite differences". Because a is a free parameter, no practitioner would impose the condition $a \geqslant a_{\mathrm{FD}}$, but instead, an optimal choice for this parameter is sought and this can be a value $a < a_{\mathrm{FD}}$.

Second, "Proof" states in its abstract that "For $a < a_{\mathrm{FD}}$, the DSC errors are less than finite differences for k near the aliasing limit, but much, much worse for smaller k". Although here "Proof" does not specify what is meant by "small k" and what is the remainder, it gives two intervals $|K| < \frac{\pi}{2}$ and $|K| > \frac{\pi}{2}$ in Section 6, where $K = kh$. In Fig. 1a, a counterexample is given to show that the DSC method outperforms the finite difference

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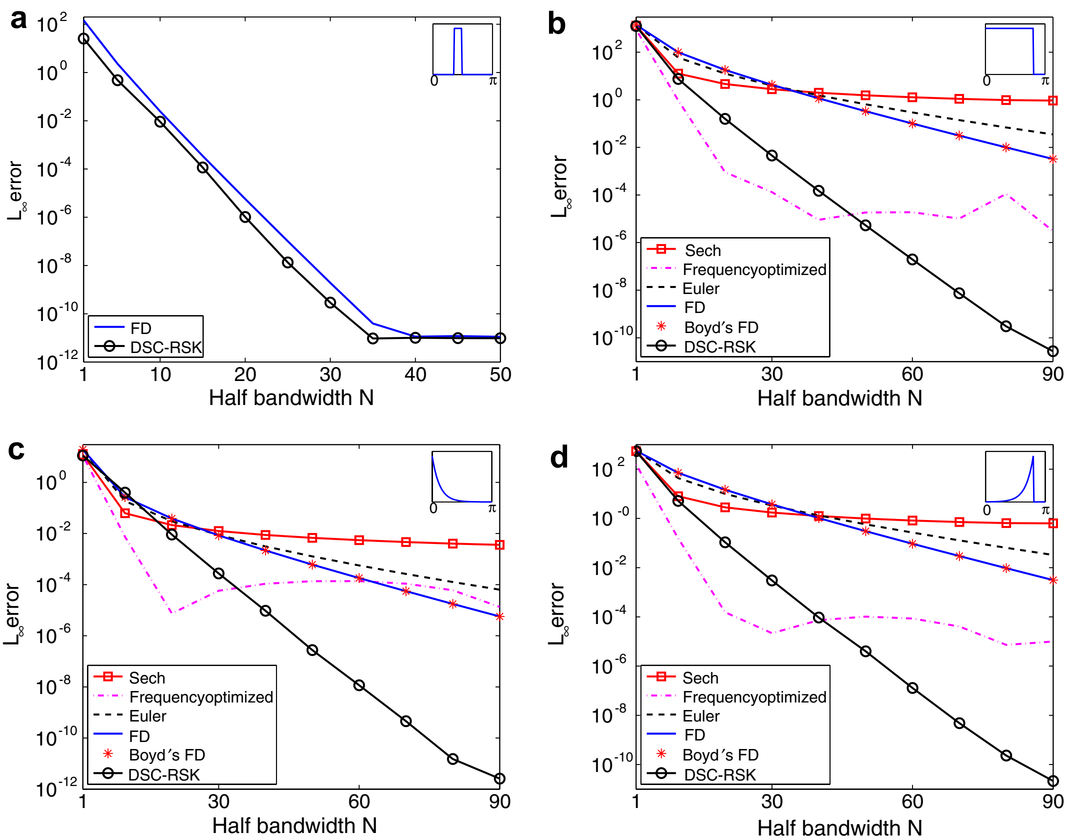

Fig. 1. Errors in differentiating $\exp(\mathrm{i}k\,x)$ with $x \in [0,2\pi]$ and $h = \pi/100$ (the Fourier frequency response is shown in a small window inside each figure). (a) Low frequency: $\sum_{j=36}^{48} \exp(\mathrm{i}jx)$; (b) wide-band frequency: $\sum_{j=0}^{80} \exp(\mathrm{i}jx)$; (c) wide-band frequency with exponentially decaying amplitude: $\sum_{j=0}^{80} \exp(-0.08j)\exp(\mathrm{i}jx)$; (d) high frequency: $\sum_{j=0}^{80} \exp(-0.08(80-j))\exp(\mathrm{i}jx)$.

(FD) method at small wavenumbers up to $K = \frac{12}{25}\pi$, over a wide range of stencils. In fact, this result was partially implied in Fig. 1 of "Proof", plotted with $N = 16$ (The vertical label in "Proof"'s Fig. 1 is incorrect.).

Third, "Proof" claims in its abstract that "Except for the very unusual case of low-pass filtered functions, that is, functions with negligible amplitude in small wavenumbers k, the DSC/LDAF is less accurate than finite differences for all stencil widths N". Fig. 1b and c show cases whose amplitudes in small wavenumbers k are either as large as or exponentially larger than those in large wavenumbers k. The DSC-RSK method outperforms the FD method up to 10^8 and 10^6 times, respectively for these two cases. Contrary to the "Proof", the DSC-RSK method is more accurate than the FD method over all stencil widths examined.

Finally, for large wavenumbers, while admitting the better performance of the DSC-RSK method than that of the standard FD scheme, "Proof" claims in its abstract that "So-called "spectrally-weighted" or "frequency-optimized" differences are superior for this special case". We therefore examine the ground of this specific claim. In Section 6, "Proof" states that "Although we shall not perform detailed comparisons between DSC and spectrally-weighted differences, the good performance of DSC for high k and small a is an accident. It seems likely that for $f(x)$ which are known to have spectra concentrated between $K = \pi/2$ and $K = \pi$, one could obtain higher accuracy from spectrally-weighted differences than from DSC". It is seen that "Proof" did not provide any detailed comparisons between the DSC and spectrally-weighted differences. Therefore, "Proof" has to present its claim as a speculation: "It seems likely that…". This speculation is turned into an affirmative conclusion that "Consequently, DSC/LDAF methods are *never* the best way to approximate derivatives on a stencil of a given width" in its abstract. Since "Proof" 's claim about spectrally-weighted differences is very general without specifying a preferred weight, logically it is sufficient for us to analyze this general claim by considering two special cases, Boyd's sech weight (Sech) [2] and frequency-optimized weight giving by the square of the Fourier transform of the function of interest (Frequency-optimized) [2,3]. It is shown in Fig. 1d that for differentiating a function with wavenumbers up to $K = \frac{4}{5}\pi$ and exponentially small amplitude in small wavenumbers k, the DSC-RSK method outperforms Boyd's two spectrally-weighted differences by many orders of magnitude.

The counterexamples given in Fig. 1 are a very small fraction of counterexamples we know. More counterexamples can be found elsewhere [10]. The essential reason for the existence of so many counterexamples is that no part of "Proof"'s central claims is based on rigorous error analysis. Error analysis for the DSC-RSK method was given earlier by Qian [6], and was not mentioned in "Proof". Asymptotic arguments are used in "Proof"'s to produce error estimates, such as those given in Eqs. (34) and (35), for the DSC-RSK method. However, these error estimates were not directly compared with those of the FD method to provide any error inequality under the conditions claimed by "Proof". In fact, Eqs. (34) and (35) indicate exponential decaying errors of the DSC-RSK under certain conditions, and are inconsistent with "Proof"'s claims. Moreover, "Proof"'s claims were not based on solid numerical evidence – only four stencil width parameters ($N = 2$, 4, 8 and 16) were examined, while "Proof"'s claims are for all possible N values. Furthermore, "Proof"'s claim about the superiority of spectrally-weighted differences over the DSC-RSK method was not supported by any analysis or numerical comparison. In the rest of this Letter, we examine "Proof"'s other major claims and present some brief remarks.

In Section 1, "Proof" states that "Since the finite difference weights are *nearly* Gaussian, one cannot escape the conclusion that the LDAF/DSC methods are really just high order finite difference methods in disguise!" In fact, the relationship between DSC and finite difference was spelled out many times in the DSC literature [9]. Although the DSC-collocation schemes can be cast into the finite difference form, a Galerkin formulation of the DSC method was also given [5,9].

In Section 1, "Proof" states that "The LDAF/DSC is also a special case of Boyd's earlier theory of sum-accelerated pseudospectral methods: special in that the weighting function is a Gaussian." As discrete approximations of singular convolution kernels, including kernels of delta type, Abel type and Hilbert type [8], the DSC has little to do with Boyd's sum-accelerated method. Moreover, the DSC-RSK approximation of derivatives does not fit into Boyd's sum-acceleration form $\frac{du}{dx} \approx \sum_{j=-N}^{N} w_{Nj}^{\mathrm{DSC}} d_j^{(1),\mathrm{sinc}}$ in general because its works both on and off grids [10].

In Section 3, "Proof" states that "However, the problem of summing slowly convergent series is an ancient one. A broad collection of schemes, known variously as "summability", "sequence acceleration" or "sum-acceleration" methods have been developed. Boyd was the first to apply such ideas to pseudospectral series to invent the form of nonstandard differences called "sum-accelerated pseudospectral"." It is to point out that the general idea of accelerating the Whittaker-Shannon-Kotel'nikov sampling, i.e., the sinc pseudospectral series, with a weight function was introduced as early as 1919 by Theis [7]. Since "Proof" devotes respectively Section 3 and Section 4 to advocate the Euler-accelerated sinc algorithm (Euler) [2] and the sum-acceleration method (denoted as Boyd's FD) [2], it is necessary to analyze the performance of these methods. As shown in Fig. 1, Boyd's FD behaves identically to the FD, while Boyd's Euler method behaves very similar to the FD. The DSC-RSK outperforms all these methods by many orders of magnitude for the problems studied.

It is true that the DSC-RSK has a free parameter a and therefore is not as robust as the FD method in practical applications, although there are ways to select appropriate a values for a given problem. The near optimal parameters for the Sech and DSC-RSK methods are chosen as the follows: In Fig. 1a, $(N,a) = (1,0.589)$, $(5,0.372)$, $(10,0.262)$, $(15,0.228)$, $(20,0.196)$, $(25,0.177)$, $(30,0.161)$, $(35,0.147)$, $(40,0.144)$, $(45,0.139)$, and $(50,0.124)$ for the DSC-RSK. In Fig. 1b–d, $D = 0.26$ for the Sech, while for the DSC-RSK, $(N,a) = (1,0.786)$, $(10,0.186)$, $(20,0.129)$, $(30,0.104)$, $(40,0.090)$, $(50,0.080)$, $(60,0.073)$, $(70,0.067)$, $(80,0.063)$, and $(90,0.058)$. All methods are optimally implemented. More details of the implementation can be found elsewhere [10]. It is emphasized that all parameters, namely a, N and k values, used in this Letter are within the parameter range claimed by "Proof".

While we demonstrate that the DSC-RSK method outperforms the FD method for differentiating $e^{(ikx)}$ with fairly small wavenumbers in Fig. 1a, it is true that the FD method is superior for functions with very small wavenumbers. It is for this reason that the DSC-RSK method was not advocated as another finite difference scheme, but as a local spectral method, to be used for problems that are difficult for both low order methods and global spectral methods. High frequency problems in structural vibration [5] and electromagnetic waves

[1] are typical examples. The DSC method had hardly been used in small stencils in its applications, except for a few cases in complying with referees' requests.

The sole purpose of this Letter is to demonstrates that central claims in ''Proof'' are unfounded. We have no intention to claim the superiority of the DSC-RSK algorithm. Given the great diversity of problems with different physical origins, we believe it is improper to claim that one method is superior to others in general without detailed analysis and comparison. However, for a given problem, one might show that some methods are more suitable than others. In view of the fact that detailed comparisons between the DSC-RSK method and many other numerical methods have not been made, it is entirely possible to find another method that outperforms the DSC-RSK method for some examples studied in this Letter and elsewhere [10]. It is also possible to come up with other examples for which the DSC-RSK method does not perform as well as other five methods examined in this Letter.

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