CLASSIFICATION OF ISING VECTORS IN THE VERTEX OPERATOR ALGEBRA V_L^+

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Abstract. Let L be an even lattice without roots. In this article, we classify all Ising vectors in the vertex operator algebra V_L^+ associated with L.

Introduction

In vertex operator algebra (VOA) theory, the simple Virasoro VOA $L(1/2,0)$ of central charge $1/2$ plays important roles. In fact, for each embedding, an automorphism, called a τ-involution, is defined using the representation theory of $L(1/2,0)$ ([Mi96]). This is useful for the study of the automorphism group of a VOA. For example, this construction gives a one-to-one correspondence between the set of subVOAs of the moonshine VOA isomorphic to $L(1/2,0)$ and that of elements in certain conjugacy class of the Monster ([Mi96, Hö10]).

Many properties of τ-involutions are studied using Ising vectors, weight 2 elements generating $L(1/2,0)$. For example, the 6-transposition property of τ-involutions was proved in [Sa07] by classifying the subalgebra generated by two Ising vectors. Hence it is natural to classify Ising vectors in a VOA. For example, this was done in [La99, LSY07] for code VOAs. However, in general, it is hard to even find an Ising vector.

Let L be an even lattice and V_L the lattice VOA associated with L. Then the subspace V_L^+ fixed by a lift of the -1-isometry of L is a subVOA of V_L. There are two constructions of Ising vectors in V_L^+ related to sublattices of L isomorphic to $\sqrt{2}A_1$ ([DMZ94]) and $\sqrt{2}E_8$ ([DLMN98, Gr98]).

The main theorem of this article is the following:

Theorem 2.5. *Let L be an even lattice without roots and e an Ising vector in V_L^+. Then there is a sublattice U of L isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$ such that $e \in V_U^+$.*

We note that this theorem was conjectured in [LSY07] and that if $L/\sqrt{2}$ is even and if L is the Leech lattice, then this theorem was proved in [LSY07] and in [LS07], respectively.

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We also note that if L has roots then the automorphism group of V_L^+ is infinite, and V_L^+ may have infinitely many Ising vectors.

In this article, we prove Theorem 2.5, and hence we classify all Ising vectors in V_L^+. Our result shows that the study of τ-involutions of V_L^+ is essentially equivalent to that of sublattices of L isomorphic to $\sqrt{2}E_8$ (cf. [GL11, GL12]).

The key is to describe the action of the τ-involution on the Griess algebra B of V_L^+. Let e be an Ising vector in V_L^+ and $L(4;e)$ the norm 4 vectors in L which appear in the description of e with respect to the standard basis of $(V_L^+)_2$ (see Section 2 for the definition of $L(4;e)$). By [LS07], the τ-involution τ_e associated to e is a lift of an automorphism g of L. We show in Lemma 2.1 that g is trivial on $\{\{\pm v\} \mid v \in L(4;e)\}$. This lemma follows from the decomposition of B with respect to the adjoint action of e ([HLY12]), the action of τ_e on it ([Mi96]) and the explicit calculations on the Griess algebra ([FLM88]). By this lemma, we can obtain a VOA V containing e on which τ_e acts trivially. By [LSY07] e is fixed by the group A generated by τ-involutions associated to elements in $L(4;e)$. Hence e belongs to the subVOA V^A of V fixed by A. Using the explicit action of A, we can find a lattice N satisfying $e \in V_N^+$ and $N/\sqrt{2}$ is even. This case was done in [LSY07].

1. Preliminaries

1.1. VOAs associated with even lattices. In this subsection, we review the VOAs V_L and V_L^+ associated with even lattice L of rank n and their automorphisms. Our notation for lattice VOAs here is standard (cf. [FLM88]).

Let L be a (positive-definite) even lattice with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. Let $H = \mathbb{C}\otimes_{\mathbb{Z}} L$ be an abelian Lie algebra and $\hat{H} = H\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$ its affine Lie algebra. Let $\hat{H}^- = H\otimes t^{-1}\mathbb{C}[t^{-1}]$ and let $S(\hat{H}^-)$ be the symmetric algebra of $\hat{H}^-$. Then $M_H(1) = S(\hat{H}^-) \cong \mathbb{C}[h(m) \mid h\in H, m<0]\cdot\mathbf{1}$ is the unique irreducible $\hat{H}$-module such that $h(m)\cdot\mathbf{1} = 0$ for $h \in H$, $m \geq 0$ and $c = 1$, where $h(m) = h\otimes t^m$. Note that $M_H(1)$ has a VOA structure.

The twisted group algebra $\mathbb{C}\{L\}$ can be described as follows. Let $\langle\kappa\rangle$ be a cyclic group of order 2 and $1 \to \langle\kappa\rangle \to \hat{L} \to L \to 1$ a central extension of L by $\langle\kappa\rangle$ satisfying the commutator relation $[e^\alpha, e^\beta] = \kappa^{\langle\alpha,\beta\rangle}$ for $\alpha,\beta\in L$. Let $L\to\hat{L}, \alpha\mapsto e^\alpha$ be a section and $\varepsilon(,) : L\times L\to\langle\kappa\rangle$ the associated 2-cocycle, that is, $e^\alpha e^\beta = \varepsilon(\alpha,\beta)e^{\alpha+\beta}$. We may assume that $\varepsilon(\alpha,\alpha) = \kappa^{\langle\alpha,\alpha\rangle/2}$ and $\varepsilon(,)$ is bilinear by [FLM88, Proposition 5.3.1]. The twisted group algebra is defined by

$$\mathbb{C}\{L\} \cong \mathbb{C}[\hat{L}]/(\kappa+1) = \mathrm{Span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in L\},$$

where $\mathbb{C}[\hat{L}]$ is the usual group algebra of the group $\hat{L}$. The lattice VOA V_L associated with L is defined to be $M_H(1)\otimes\mathbb{C}\{L\}$ ([Bo86, FLM88]).

For any sublattice E of L, let $\mathbb{C}\{E\} = \mathrm{Span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in E\}$ be a subalgebra of $\mathbb{C}\{L\}$ and let $H_E = \mathbb{C} \otimes_{\mathbb{Z}} E$ be a subspace of $H = \mathbb{C} \otimes_{\mathbb{Z}} L$. Then the subspace $S(\hat{H}_E^-) \otimes \mathbb{C}\{E\}$ forms a subVOA of V_L and it is isomorphic to the lattice VOA V_E.

Let $O(\hat{L})$ be the subgroup of $\mathrm{Aut}(\hat{L})$ induced from $\mathrm{Aut}(L)$. By [FLM88, Proposition 5.4.1] there is an exact sequence of groups

$$1 \to \mathrm{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \to O(\hat{L}) \xrightarrow{-} \mathrm{Aut}(L) \to 1.$$

Note that for $f \in O(\hat{L})$

$$f(e^\alpha) \in \{\pm e^{\bar{f}(\alpha)}\}. \tag{1.1}$$

By [FLM88, Corollary 10.4.8], $f \in O(\hat{L})$ acts on V_L as an automorphism as follows:

$$f(h_{i_1}(n_1)h_{i_2}(n_2)\dots h_{i_k}(n_k) \otimes e^\alpha) = \bar{f}(h_{i_1})(n_1)\bar{f}(h_{i_2})(n_2)\dots \bar{f}(h_{i_k})(n_k) \otimes f(e^\alpha), \tag{1.2}$$

where $n_i \in \mathbb{Z}_{<0}$ and $\alpha \in L$. Hence $O(\hat{L})$ is a subgroup of $\mathrm{Aut}(V_L)$.

Let θ be the automorphism of $\hat{L}$ defined by $\theta(e^\alpha) = e^{-\alpha}$, $\alpha \in L$. Then $\bar{\theta} = -1 \in \mathrm{Aut}(L)$. Using (1.2) we view θ as an automorphism of V_L. Let $V_L^+ = \{v \in V_L \mid \theta(v) = v\}$ be the subspace of V_L fixed by θ. Then V_L^+ is a subVOA of V_L. Since θ is a central element of $O(\hat{L})$, the quotient group $O(\hat{L})/\langle\theta\rangle$ is a subgroup of $\mathrm{Aut}(V_L^+)$. Note that V_L^+ is a simple VOA of CFT type.

Later, we will consider the subVOA of V_L^+ generated by the weight 2 subspace.

Lemma 1.1. (cf. [FLM88, Proposition 12.2.6]) *Let L be an even lattice without roots. Let N be the sublattice of L generated by $L(4)$. Then the subVOA of V_L^+ generated by $(V_L^+)_2$ is $(V_N \otimes M_{H'}(1))^+$, where $H' = (\langle N\rangle_{\mathbb{C}})^\perp$ in $\langle L\rangle_{\mathbb{C}}$.*

1.2. **Ising vectors and τ-involutions.** In this subsection, we review Ising vectors and corresponding τ-involutions.

Definition 1.2. A weight 2 element e of a VOA is called an *Ising vector* if the vertex subalgebra generated by e is isomorphic to the simple Virasoro VOA of central charge 1/2 and e is its conformal vector.

For an Ising vector e, the automorphism τ_e, called the *τ-involution* or *Miyamoto involution*, was defined in ([Mi96, Theorem 4.2]) based on the representation theory of the simple Virasoro VOA of central charge 1/2 ([DMZ94]).

Let V be a VOA of CFT type with $V_1 = 0$. Then the first product $(a, b) \mapsto a \cdot b = a_{(1)}b$ provides a (nonassociative) commutative algebra structure on V_2. This algebra V_2 is called the *Griess algebra* of V. The action of τ_e on the Griess algebra was described in [HLY12] as follows:

Lemma 1.3. [HLY12, Lemma 2.6] *Let V be a simple VOA of CFT type with $V_1 = 0$ and e an Ising vector in V. Then $B = V_2$ has the following decomposition with respect to the adjoint action of e:*

$$B = \mathbb{C}e \oplus B^e(0) \oplus B^e(1/2) \oplus B^e(1/16),$$

where $B^e(k) = \{v \in B \mid e \cdot v = kv\}$. Moreover, the automorphism τ_e acts on B as follows:

$$1 \quad \text{on} \quad \mathbb{C}e \oplus B^e(0) \oplus B^e(1/2), \qquad -1 \quad \text{on} \quad B^e(1/16).$$

In the proof of the main theorem, we need the following lemma:

Lemma 1.4. [LSY07, Lemma 3.7] *Let V be a VOA of CFT type with $V_1 = 0$. Suppose that V has two Ising vectors e, f and that $\tau_e = \mathrm{id}$ on V. Then e is fixed by τ_f, namely $e \in V^{\tau_f}$.*

Let L be an even lattice of rank n without roots, that is, $L(2) = \{v \in L \mid \langle v, v\rangle = 2\} = \emptyset$. Then $(V_L^+)_1 = 0$, and we can consider the Griess algebra $B = (V_L^+)_2$ of V_L^+. Let $\{h_i \mid 1 \le i \le n\}$ be an orthonormal basis of $H = \mathbb{C} \otimes_{\mathbb{Z}} L = \langle L\rangle_{\mathbb{C}}$. Set $L(4) = \{v \in L \mid \langle v, v\rangle = 4\}$. For $1 \le i \le j \le n$ and $\alpha \in L(4)$, set $h_{ij} = h_i(-1)h_j(-1)\mathbf{1}$ and $x_\alpha = e^\alpha + e^{-\alpha} = e^\alpha + \theta(e^\alpha)$. Note that $x_\alpha = x_{-\alpha}$.

Lemma 1.5. [FLM88, Section 8.9]

(1) *The set*

$$\{h_{ij}, x_\alpha \mid 1 \le i \le j \le n,\ \{\pm\alpha\} \subset L(4)\}$$

is a basis of B.

(2) *The products of the basis of B given in (1) are the following:*

$$\begin{aligned}
h_{ij} \cdot h_{kl} &= \delta_{ik}h_{jl} + \delta_{il}h_{jk} + \delta_{jk}h_{il} + \delta_{jl}h_{ik},\\
h_{ij} \cdot x_\alpha &= \langle h_i, \alpha\rangle\langle h_j, \alpha\rangle x_\alpha,\\
x_\alpha \cdot x_\beta &= \begin{cases} \varepsilon(\alpha,\beta)x_{\alpha\pm\beta} & \text{if } \langle\alpha,\beta\rangle = \mp 2,\\ \alpha(-1)^2\mathbf{1} & \text{if } \alpha = \pm\beta,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Let $\alpha \in L(4)$. Then the elements $\omega^+(\alpha)$ and $\omega^-(\alpha)$ of V_L^+ defined by

$$\omega^\pm(\alpha) = \frac{1}{16}\alpha(-1)^2 \cdot \mathbf{1} \pm \frac{1}{4}x_\alpha \tag{1.3}$$

are Ising vectors ([DMZ94, Theorem 6.3]). The following lemma is easy:

Lemma 1.6. *The automorphisms $\tau_{\omega^\pm(\alpha)}$ of V_L^+ act by*

$$u \otimes x_\beta \mapsto (-1)^{\langle\alpha,\beta\rangle}u \otimes x_\beta \qquad \textit{for } u \in M_H(1) \textit{ and } \beta \in L.$$

In general, the following holds:

Proposition 1.7. [LS07, Lemma 5.5] *Let L be an even lattice without roots and e an Ising vector in V_L^+. Then $\tau_e \in O(\hat{L})/\langle\theta\rangle$.*

We note that the main theorem was proved if $L/\sqrt{2}$ is even as follows:

Proposition 1.8. [LSY07, Theorem 4.6] *Let L be an even lattice and e an Ising vector in V_L^+. Assume that the lattice $L/\sqrt{2}$ is even. Then there is a sublattice U of L isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$ such that $e \in V_U^+$.*

2. Classification of Ising vectors in V_L^+

Let L be an even lattice of rank n without roots and e an Ising vector in V_L^+. Then by Lemma 1.5 (1)

$$e = \sum_{i\le j} c_{ij}^e h_{ij} + \sum_{\{\pm\alpha\}\subset L(4)} d_{\{\pm\alpha\}}^e x_\alpha, \tag{2.1}$$

where $c_{ij}^e, d_{\{\pm\alpha\}}^e \in \mathbb{C}$. Set $L(4;e) = \{\alpha \in L(4) \mid d_{\{\pm\alpha\}}^e \neq 0\}$, $H_1 = \langle L(4;e)\rangle_{\mathbb{C}}$ and $H_2 = H_1^\perp$ in H. Note that if $\alpha \in L(4;e)$ then $-\alpha \in L(4;e)$. Without loss of generality, we may assume that $h_i \in H_1$ if $1 \le i \le \dim H_1$. Then $H_2 = \mathrm{Span}_{\mathbb{C}}\{h_j \mid \dim H_1 + 1 \le j \le n\}$.

By Proposition 1.7, $\tau_e \in O(\hat{L})/\langle\theta\rangle$. Since $e \in V_L$, we regard τ_e as an automorphism of V_L. Then $\tau_e \in O(\hat{L})$, and set $g = \bar{\tau}_e \in \mathrm{Aut}(L)$. Since τ_e is of order 1 or 2, so is g. The following is the key lemma in this article:

Lemma 2.1. *Let $\beta \in L(4;e)$. Then $g(\beta) \in \{\pm\beta\}$.*

Proof. By (1.1) and (1.2),

$$\tau_e(x_\beta) \in \{\pm x_{g(\beta)}\}. \tag{2.2}$$

On the other hand, $\tau_e(e) = e$, (1.2) and (2.1) show

$$\tau_e(d_{\{\pm\beta\}}^e x_\beta) = d_{\{\pm g(\beta)\}}^e x_{g(\beta)}. \tag{2.3}$$

By (2.2) and (2.3),

$$\frac{d_{\{\pm g(\beta)\}}^e}{d_{\{\pm\beta\}}^e} \in \{\pm 1\}. \tag{2.4}$$

Suppose $g(\beta) \notin \{\pm\beta\}$. Then $x_\beta - \tau_e(x_\beta)$ is non-zero, and it is an eigenvector of τ_e with eigenvalue -1. By Lemma 1.3, we have

$$e \cdot (x_\beta - \tau_e(x_\beta)) = \frac{1}{16}(x_\beta - \tau_e(x_\beta)). \tag{2.5}$$

Let us calculate the image of both sides of (2.5) under the canonical projection $\mu : (V_L^+)_2 \to \mathrm{Span}_{\mathbb{C}}\{h_{ij} \mid 1 \le i \le j \le n\}$ with respect to the basis given in Lemma 1.5 (1). By (2.2) the image of the right hand side of (2.5) under μ is 0:

$$\mu\left(\frac{1}{16}(x_\beta - \tau_e(x_\beta))\right) = 0. \tag{2.6}$$

Let us discuss the left hand side of (2.5). By Lemma 1.5 (2) and (2.4), we have

$$\begin{aligned} e \cdot (x_\beta - \tau_e(x_\beta)) &= \left(\sum_{i\le j} c^e_{ij} h_{ij} + \sum_{\{\pm\alpha\}\subset L(4)} d^e_{\{\pm\alpha\}} x_\alpha\right) \cdot (x_\beta - \tau_e(x_\beta)) \\ &\in d^e_{\{\pm\beta\}}\left(\beta(-1)^2\mathbf{1} - g(\beta)(-1)^2\mathbf{1}\right) + \mathrm{Span}_{\mathbb{C}}\{x_\gamma \mid \{\pm\gamma\} \subset L(4)\}. \end{aligned}$$

Thus

$$\begin{aligned} \mu(e \cdot (x_\beta - \tau_e(x_\beta))) &= d^e_{\{\pm\beta\}}\left(\beta(-1)^2\mathbf{1} - g(\beta)(-1)^2\mathbf{1}\right) \\ &= d^e_{\{\pm\beta\}}\left(\beta - g(\beta)\right)(-1)(\beta + g(\beta))(-1)\mathbf{1}. \end{aligned}$$

This is not zero by $g(\beta) \notin \{\pm\beta\}$, which contradicts (2.5) and (2.6). Therefore $g(\beta) \in \{\pm\beta\}$. $\square$

For $\varepsilon \in \{\pm\}$, set $L(4; e, \varepsilon) = \{v \in L(4; e) \mid g(v) = \varepsilon v\}$, $L^{e,\varepsilon} = \langle L(4; e, \varepsilon)\rangle_{\mathbb{Z}}$, and $H_1^\varepsilon = \langle L^{e,\varepsilon}\rangle_{\mathbb{C}}$. Since g preserves the inner product, $H_1 = H_1^+ \perp H_1^-$ and g acts on $H_2 = H_1^\perp$. Let $H_2^\pm$ be ±1-eigenspaces of g in H_2. For $\varepsilon \in \{\pm\}$, let W^ε be a lattice of full rank in H_2^ε isomorphic to an orthogonal direct sum of copies of $2A_1$. Then

$$M_{H_2^\varepsilon}(1) \subset V_{W^\varepsilon}. \tag{2.7}$$

Lemma 2.2. *The Ising vector e belongs to the VOA $V^+_{L^{e,+}\oplus W^+} \otimes V^+_{L^{e,-}\oplus W^-}$, and $\tau_e = \mathrm{id}$ on this VOA.*

Proof. By Lemma 2.1, $L(4; e) = L(4; e, +) \cup L(4; e, -)$. Hence, by (2.1) and (2.7),

$$e \in (V_{L^{e,+}} \otimes M_{H_2^+}(1) \otimes V_{L^{e,-}} \otimes M_{H_2^-}(1))^+ \subset V^+_{L^{e,+}\oplus W^+\oplus L^{e,-}\oplus W^-}. \tag{2.8}$$

Since g acts by ±1 on $L^{e,\pm} \oplus W^\pm$, the subspace of (2.8) fixed by τ_e is

$$V^+_{L^{e,+}\oplus W^+} \otimes V^+_{L^{e,-}\oplus W^-}.$$

Since e is fixed by τ_e, we have the desired result. $\square$

We now prove the main theorem.

Theorem 2.3. *Let L be an even lattice without roots. Let e be an Ising vector in V_L^+. Then there is a sublattice U of L isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$ such that $e \in V_U^+$.*

Proof. Set $V = V^+_{L^{e,+}\oplus W^+} \otimes V^+_{L^{e,-}\oplus W^-}$. By Lemma 2.2, e belongs to V and $\tau_e = \mathrm{id}$ on V. Let $A = \langle \tau_{\omega^\pm(\beta)} \mid \beta \in L(4;e)\rangle$. By Lemma 1.4, e belongs to the subVOA V^A of V fixed by A. Since e is a weight 2 element, it is contained in the subVOA generated by $(V^A)_2$. By Lemmas 1.1 and 1.6 and (2.7) (cf. (2.8)),

$$e \in V^+_{N^+\oplus K^+} \otimes V^+_{N^-\oplus K^-} \subset V^+_N,$$

where for $\varepsilon \in \{\pm\}$, $N^\varepsilon = \mathrm{Span}_{\mathbb{Z}}\{v \in L(4;e,\varepsilon) \mid \langle v, L(4;e)\rangle \in 2\mathbb{Z}\}$, K^ε is a lattice of full rank in $(\langle N^\varepsilon\rangle_{\mathbb{C}})^\perp \cap (H_1^\varepsilon \oplus H_2^\varepsilon)$ isomorphic to an orthogonal direct sum of copies of $2A_1$, and $N = N^+ \oplus K^+ \oplus N^- \oplus K^-$. Since N is generated by norm 4 and 8 vectors, and the inner products of the generator belong to $2\mathbb{Z}$, the lattice $N/\sqrt{2}$ is even. By Proposition 1.8, there is a sublattice U of N isomorphic to $\sqrt{2}A_1$ or $\sqrt{2}E_8$ such that $e \in V_U^+$. It follows from $K^+(4) = K^-(4) = \emptyset$ that $N(4) = N^+(4) \cup N^-(4) \subset L$. Since $\sqrt{2}A_1$ and $\sqrt{2}E_8$ are spanned by norm 4 vectors as lattices, we have $U \subset L$. Hence V_U^+ is a subVOA of V_L^+. □

As an application of the main theorem, we count the total number of Ising vectors in V_L^+ for even lattice L without roots.

Let us describe Ising vectors in V_L^+. The Ising vector $\omega^\pm(\alpha)$ associated to $\alpha \in L(4)$ was described in (1.3) as follows:

$$\omega^\pm(\alpha) = \frac{1}{16}\alpha(-1)^2 \cdot \mathbf{1} \pm \frac{1}{4}x_\alpha.$$

Let E be an even lattice isomorphic to $\sqrt{2}E_8$ and $\{u_i \mid 1 \le i \le 8\}$ an orthonormal basis of $\mathbb{C}\otimes_{\mathbb{Z}} E$. We consider the trivial 2-cocycle of $\mathbb{C}\{E\}$ for V_E. Then for $\varphi \in \mathrm{Hom}(E, \mathbb{Z}/2\mathbb{Z}) (\cong (\mathbb{Z}/2\mathbb{Z})^8)$

$$\omega(E,\varphi) = \frac{1}{32}\sum_{i=1}^{8} u_i(-1)^2 \cdot \mathbf{1} + \frac{1}{32}\sum_{\{\pm\alpha\}\subset E(4)} (-1)^{\varphi(\alpha)} x_\alpha$$

is an Ising vector in V_E^+ ([DLMN98, Gr98]). Since $E(4)$ spans E as a lattice, $\omega(E,\varphi) = \omega(E,\varphi')$ if and only if $\varphi = \varphi'$. Hence V_E^+ has 256 Ising vectors of form $\omega(E,\varphi)$. Thus $V^+_{\sqrt{2}A_1}$ and $V^+_{\sqrt{2}E_8}$ has exactly 2 and 496 Ising vectors, respectively ([LSY07, Proposition 4.2 and 4.3]).

Corollary 2.4. *Let L be an even lattice without roots. Then the number of Ising vectors in V_L^+ is given by*

$$|L(4)| + 256 \times |\{U \subset L \mid U \cong \sqrt{2}E_8\}|.$$

Proof. Set $m = |L(4)| + 256 \times |\{E \subset L \mid E \cong \sqrt{2}E_8\}|$. Theorem 2.3 shows that the number of Ising vectors in V_L^+ is less than or equal to m. Let us show that there are exactly m

Ising vectors in V_L^+, that is, the Ising vectors $\omega^{\pm}(\alpha)$ and $\omega(E,\varphi)$ are distinct. By Lemma 1.5 (1), $\omega^{\varepsilon}(\alpha)=\omega^{\delta}(\beta)$ if and only if $\alpha=\beta$ and $\varepsilon=\delta$. Moreover, $\omega^{\varepsilon}(\alpha)\neq\omega(E,\varphi)$ for all $\alpha\in L(4)$, $L\supset E\cong\sqrt{2}E_8$ and $\varphi\in\mathrm{Hom}(E,\mathbb{Z}/2\mathbb{Z})$.

Let E_1,E_2 be sublattices of L such that $E_1\cong E_2\cong\sqrt{2}E_8$. Let $\varphi_i\in\mathrm{Hom}(E_i,\mathbb{Z}/2\mathbb{Z})$, $i=1,2$. Then it follows from Lemma 1.5 (1) and $\langle E_i(4)\rangle_{\mathbb{Z}}=E_i$ that $\omega(E_1,\varphi_1)=\omega(E_2,\varphi_2)$ if and only if $E_1=E_2$ and $\varphi_1=\varphi_2$. Therefore, there are exactly m Ising vectors in V_L^+. $\square$

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