MS-EV0004 Vertex operator algebras: Lecture 11

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8 Zhu algebra

The ultimate goal of the representation theory of a vertex algebra is to determine the category of its modules. In this section, we make the first step towards the goal, but we need to narrow down our attention as well. Recall that a vertex operator algebra V is $\mathbb{Z}$-graded, but we assume that it only has non-negative degrees:

$$V = \bigoplus_{n=0}^{\infty} V_n.$$

We also focus on modules that have a non-negative grading.

In such a setting, we can define an associative algebra $A(V)$ associated to V, called the Zhu algebra of V and see that finite-dimensional simple $A(V)$-modules and simple V-modules are in one-to-one correspondence up to isomorphism.

Reference:

- Y. Zhu, "Modular invariance of characters of vertex operator algebras", J. Amer. Math. Soc. (1996).

8.1 Modules over VOAs

Let $(V, \mathbf{1}, Y, \omega)$ be a vertex operator algebra. As we have already declared, V is has a non-negative grading. Recall that we write $\deg a = n$ if $a \in V_n$. Also, for such a homogeneous element a, the corresponding operators have clear degrees

$$\deg a_{(n)} = \deg a - n - 1, \quad n \in \mathbb{Z}.$$

To define a module over such a VOA, we require that these degrees of operators are transferred to a representation space.

Definition 8.1. Let $(V, \mathbf{1}, Y, \omega)$ be a vertex operator algebra as above. An $\mathbb{N}$-graded module over V is a module (W, Y_W) over V as a vertex algebra such that the following conditions are satisfied:

- the vector space W is $\mathbb{N}$-graded:

$$W = \bigoplus_{n=0}^{\infty} W_n, \quad \dim W_n < \infty, \, n \in \mathbb{N}, \quad W_0 \neq 0,$$

- with respect to this grading, each operator $a^W_{(n)}$ has the degree

$$\deg a^W_{(n)} = \deg a - n - 1,$$

for a homogeneous $a \in V$.

If $M = \bigoplus_{n=0}^{\infty} M_n$ and $N = \bigoplus_{n=0}^{\infty} N_n$ are $\mathbb{N}$-graded V-modules, a morphism $f\colon M \to N$ is a linear map with the following properties:

- for all $a \in V$, $w \in M$, $n \in \mathbb{Z}$,

$$f(a^M_{(n)} w) = a^N_{(n)} f(w).$$

- there is $k \in \mathbb{Z}$ such that $f(M_n) \subset N_{n+k}$ for all $n \in \mathbb{Z}$.

The image of a morphism $f\colon M \to N$ is called a **submodule** of N (so the category of $\mathbb{N}$-graded modules is closed under quotient). We say that an $\mathbb{N}$-graded V-module $M(\neq 0)$ is **simple** if its submodule is either 0 or M.

By the Jacobi identity, when we set $L^M_n = \omega^M_{(n+1)}$, $n \in \mathbb{Z}$, they form a representation of the Virasoro algebra on M, but the grading parameter of M is a priori nothing to do with the eigenvalues of L^M_0, and we do not even require that L^M_0 is diagonalizable in contrast to the VOA itself. However, for a simple module, the grading something to do with the eigenvalues of L^M_0.

Theorem 8.2. *Let $M = \bigoplus_{n=0}^{\infty} M_n$ be a simple $\mathbb{N}$-gradable V-module. Then, there exists $h \in \mathbb{C}$ such that*

$$L^M_0|_{M_n} = (h+n)\mathrm{Id}_{M_n}, \quad n \in \mathbb{Z}.$$

Proof. ($\to$ Exercise.) □

Example 8.3. Let $(\mathcal{F}_0, \mathbf{1}, Y, \omega)$ be the Heisenberg VOA. Recall that the pair $(\mathcal{F}_\lambda, Y_{\mathcal{F}_\lambda})$ for each $\lambda \in \mathbb{C}$ is a module over $\mathcal{F}_0$ as a vertex algebra. For each $n \in \mathbb{N}$, let us set

$$(\mathcal{F}_\lambda)_n = \mathrm{Span}\Big\{\alpha_{-n_1} \cdots \alpha_{-n_l} |\lambda\rangle \,\Big|\, \sum_{i=1}^{l} n_i = n\Big\}.$$

Then, the pair $(\mathcal{F}_\lambda = \bigoplus_{n=0}^{\infty} (\mathcal{F}_\lambda)_n, Y_{\mathcal{F}_\lambda})$ is a simple $\mathbb{N}$-gradable $\mathcal{F}_0$-module.

In fact, from the Jacobi identity, we get

$$[L^{\mathcal{F}_\lambda}_0, a^{\mathcal{F}_\lambda}_{(n)}] = (\deg a - n - 1) a^{\mathcal{F}_\lambda}_{(n)}$$

for all homogeneous $a \in V$ and $n \in \mathbb{Z}$. This proves

$$L^{\mathcal{F}_\lambda}_0|_{(\mathcal{F}_\lambda)_n} = \left(\frac{\lambda^2}{2} + n\right) \mathrm{Id}_{\mathcal{F}_\lambda}, \quad n \in \mathbb{N}$$

and

$$\deg a^{\mathcal{F}_\lambda}_{(n)} = \deg a - n - 1$$

for a homogeneous $a \in V$ and $n \in \mathbb{Z}$.

A submodule $N \subset \mathcal{F}_\lambda$ is preserved by all operators $a^{\mathcal{F}_\lambda}_{(n)}$, $a \in V$, $n \in \mathbb{Z}$, in particular by $\widehat{\mathfrak{h}}$. Since $\mathcal{F}_\lambda$ is an irreducible representation of $\widehat{\mathfrak{h}}$ (Exercise 1.2.3), it must be a simple $\mathcal{F}_0$-module as well.

8.2 Zhu algebra and modules

Definition 8.4. Define bilinear maps

$$*\colon V \times V \to V, \quad \circ\colon V \times V \to V$$

by

$$\begin{aligned} a * b &= [x^{-1}] \left(Y(a,x) \frac{(x+1)^{\deg a}}{x} b \right), \\ a \circ b &= [x^{-1}] \left(Y(a,x) \frac{(x+1)^{\deg a}}{x^2} b \right) \end{aligned}$$

for homogeneous $a \in V$ and $b \in V$.

Note that $(x+1)^{\deg a}$ is a polynomial in x under our assumption that V only has non-negative degrees.

Theorem 8.5. *Let us set $O(V) = \{a \circ b | a, b \in V\}$ and $A(V) = V/O(V)$. Then, we have the following.*

(1) $(A(V), *)$ is an associative algebra, i.e.,

$$\begin{aligned} &O(V) * V \subset O(V), \quad V * O(V) \subset O(V), \\ &a * (b * c) - (a * b) * c \in O(V), \quad a, b, c \in V. \end{aligned}$$

(2) $[\mathbf{1}] = \mathbf{1} + O(V)$ is the unit of $(A(V), *)$.

(3) $[\omega] = \omega + O(V)$ is a central element of $(A(V), *)$.

Definition 8.6. The associative algebra $(A(V), *)$ is called the Zhu algebra of V.

Theorem 8.7. *Let $M = \bigoplus_{n=0}^{\infty} M_n$ be an $\mathbb{N}$-gradable V-module. For a homogeneous $a \in V$, we write $o(a) = a^M_{(\deg a - 1)}$ and extend the symbol linearly. Then,*

$$V \to \mathrm{End}(M_0); \quad a \mapsto o(a)$$

induces an action of $A(V)$ on M_0, i.e., $o(a)|_{M_0} = 0$ for all $a \in O(V)$ and $o(a * b)|_{M_0} = o(a)o(b)|_{M_0}$ for $a, b \in V$.

Theorem 8.8. *Let W be a finite dimensional $A(V)$-module. Then, there exists an $\mathbb{N}$-gradable V-module $M = \bigoplus_{n=0}^{\infty} M_n$ with the following properties.*

(1) $M_0 \simeq W$ as $A(V)$-modules.

(2) If $N \subset M$ is a submodule such that $N \cap M_0 = 0$, then $N = 0$.

Theorem 8.9. *Theorems 8.7 and 8.8 induce a one-to-one correspondence between finite-dimensional simple $A(V)$-modules and simple $\mathbb{N}$-gradable V-modules.*

Sketch of Proof. Let W be a finite-dimensional $A(V)$-module. If we apply Theorem 8.8, we get an $\mathbb{N}$-gradable V-module $M = \bigoplus_{n=0}^{\infty} M_n$ with $M_0 \simeq W$ as $A(V)$-modules. Furthermore, if W is simple, then so is M.

Conversely, let us start with an $\mathbb{N}$-gradable V-module $M = \bigoplus_{n=0}^{\infty} M_n$. If we apply Theorem 8.8 to M_0, we get an $\mathbb{N}$-gradable V-module $\widetilde{M} = \bigoplus_{n=0}^{\infty} \widetilde{M}_n$ with $\widetilde{M}_0 \simeq M_0$ as $A(V)$-modules, but we cannot say that $\widetilde{M} \simeq M$. However, if M is simple, then M_0 is simple as a $A(V)$-module and $\widetilde{M}$ is also a simple module. Furthermore, we can construct a non-zero morphism $\widetilde{M} \to M$, but since they are irreducible, we must have $\widetilde{M} \simeq M$. □

8.3 Example: Heisenberg VOA

Theorem 8.10. *We have an isomorphism of associative algebras*

$$\mathbb{C}[\alpha] \xrightarrow{\sim} A(\mathcal{F}_0); \quad \alpha \mapsto [\alpha_{-1}\mathbf{1}].$$

Proof. See

- Frenkel–Zhu, "Vertex operator algebras associated to representations of affine and Virasoro algebras", Duke Math. J. (1992).

(The Heisenberg algebra is an affine Lie algebra.) □

Since $\mathbb{C}[\alpha]$ is commutative, finite-dimentional simple $\mathbb{C}[\alpha]$-modules are all one-dimensional. They are labelled by $\lambda \in \mathbb{C}$: for each $\lambda \in \mathbb{C}$, the one-dimensional space $\mathbb{C}_\lambda = \mathbb{C}$ is equipped with the action of $\mathbb{C}[\alpha]$,

$$\alpha \mapsto \lambda \cdot \mathrm{Id}.$$

On the other hand, $[\alpha_{-1}\mathbf{1}] \in A(\mathcal{F}_0)$ acts on $(\mathcal{F}_\lambda)_0 = \mathbb{C}\,|\lambda\rangle$ by $\lambda{\cdot}\mathrm{Id}$. Therefore, $\mathbb{C}_\lambda$ and $\mathcal{F}_\lambda$ correspond to each other under Theorem 8.9.

8.4 Example: Virasoro VOA

8.4.1 Universal Virasoro VOA

Recall that the Virasoro VOA of central charge c is built on

$$V_c = \mathcal{U}(\mathfrak{vir})/\Big(\sum_{n\geq -1} \mathcal{U}(\mathfrak{vir})L_n + \mathcal{U}(\mathfrak{vir})(C-c)\Big)$$

and the Verma modules

$$M(c,h) = \mathcal{U}(\mathfrak{vir})/\Big(\sum_{n>0} \mathcal{U}(\mathfrak{vir})L_n + \mathcal{U}(\mathfrak{vir})(L_0-h) + \mathcal{U}(\mathfrak{vir})(C-c)\Big)$$

are modules over V_c as a vertex algebra. It is not difficult to see that they are $\mathbb{N}$-gradable V_c-modules, but are not necessarily simple (in contrast to the Heisenberg case). In fact, for a certain choice of (c,h), the Verma module $M(c,h)$ is reducible as a representation of the Virasoro algebra, hence cannot be simple as V_c-module. (The V_c action goes through the action of the Virasoro algebra.) Nevertheless, we can always take the simple quotient of $M(c,h)$ and write it as $L(c,h)$.

Theorem 8.11. *There is an isomorphism of associative algebras*

$$\mathbb{C}[\mathsf{h}] \xrightarrow{\sim} A(V_c); \quad \mathsf{h} \mapsto [\omega].$$

Proof. See Frenkel–Zhu (1992), or

- W. Wang, "Rationality of Virasoro vertex operator algebras" International Mathematics Research Notices (1993).

□

Exactly by the same reasoning as the Heisenberg case, we can conclude that $L(c,h)$, $h \in \mathbb{C}$ are all of the simple $\mathbb{N}$-gradable V_c-modules.

8.4.2 Minimal Virasoro VOA

The representation V_c of the Virasoro algebra is not always irreducible, or in other words, it is not a simple VOA. It is known from the representation theory of the Virasoro algebra, when we set

$$c = c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}, \quad p,q \in \{2,3,\dots\} : \text{coprime},$$

V_c is reducible, and its maximal proper submodule is generated by a singular vector $v_{p,q} \in (V_c)_{(p-1)(q-1)}$. For this fact, see

- Iohara–Koga, "Representation theory of the Virasoro algebra", (2011).

Let us write L_c for the simple quotient. As the formula of $c_{p,q}$ is symmetric under exchanging p and q, we may assume $p < q$. If $(p,q) = (2,3)$, $v_{p,q}$ lives in $(V_c)_2$, which is spanned by $L_{-2}\mathbf{1}_c$ and L_c is the trivial representation. Usually, we exclude this case. The submodule generated by $v_{p,q}$ is also an ideal of the VOA V_c, hence L_c is still a VOA. The simple VOA L_c with some $c = c_{p,q}$ is called a minimal Virasoro VOA.

From the general theory of the Zhu algebra, the Zhu algebra $A(L_c)$ is the following quotient of $A(V_c)$:

$$A(V_c)/([v_{p,q}]).$$

It can be shown (Wang, 1993) that, under the identification $\mathbb{C}[\mathsf{h}] \simeq A(V_c)$, we can take a polynomial $G_{p,q}(\mathsf{h})$ of degree $\frac{1}{2}(p-1)(q-1)$ as a generator of the ideal.

Theorem 8.12 (Wang). *The following polynomial works as a generator of the ideal:*

$$G_{p,q}(\mathsf{h}) = \Big(\prod_{r=1}^{p-1}\prod_{s=1}^{q-1}(\mathsf{h} - h_{r,s})\Big)^{1/2},$$

where

$$h_{r,s} = \frac{(sp-rq)^2 - (p-q)^2}{4pq}, \quad r,s \in \mathbb{Z}.$$

Exercise 8.1. Show that $G_{p,q}(\mathsf{h})$ is a polynomial.

Corollary 8.13. *The minimal Virasoro VOA $L_{c_{p,q}}$ has $\frac{1}{2}(p-1)(q-1)$ number of simple $\mathbb{N}$-gradable modules, and they are covered by*

$$L(c_{p,q}, h_{r,s}), \quad r = 1, \dots, p-1, 1 \le s < \frac{q}{p}.$$

Example 8.14 ($(p,q) = (3,4)$). The central charge is $c = \frac{1}{2}$. The VOA L_c itself is a simple $\mathbb{N}$-gradable module, so we of course have

$$h_{1,1} = h_{2,3} = 0.$$

The other non-trivial modules have the conformal weights

$$h_{2,1} = h_{1,3} = \frac{1}{2} \quad \text{and} \quad h_{2,2} = h_{1,2} = \frac{1}{16}.$$

Consequently, the polynomial $G_{3,4}(\mathsf{h})$ is

$$G_{3,4}(\mathsf{h}) = \mathsf{h}\Big(\mathsf{h} - \frac{1}{2}\Big)\Big(\mathsf{h} - \frac{1}{16}\Big).$$

8.5 Proof of Theorem 8.8 (Sketch)

It is instructive to see the following for an $\mathbb{N}$-gradable V-module. Let (M, Y_M) be an $\mathbb{N}$-gradable V-module. We suppress M from the notation if there is no confusion. According to the direct sum $M = \bigoplus_{n=0}^{\infty} M_n$, a functional $\varphi \in M_0^*$ is naturally extended to the whole M.

Lemma 8.15. *For any $a^1, \ldots, a^n \in V$, $w \in M_0$, and $\varphi \in M_0^*$, we have*

$$\begin{aligned}
&\langle \varphi, Y(a^1, x_1) Y(a^2, x_2) \ldots Y(a^n, x_n) w \rangle \\
&= \langle o(a^1)^* \varphi, Y(a^2, x_2) \ldots Y(a^n, x_n) w \rangle \\
&+ \sum_{k=2}^{n} \sum_{i=0}^{\infty} \iota_{1k} F_{\deg a^1, i}(x_1, x_k) \cdot \left\langle \varphi, Y(a^2, x_2) \cdots Y(a^1_{(i)} a^k, x_k) \cdots Y(a^n, x_n) w \right\rangle,
\end{aligned}$$

where

$$F_{n,i}(x, y) = x^{-n} \partial_y^{(i)} \frac{y^n}{x - y} \in \mathbb{C}[x, y][x^{-1}, y^{-1}, (x - y)^{-1}]$$

for $n, i \in \mathbb{N}$.

Proof. ($\to$ Exercise) $\square$

Notice that, in this formula, the right-most M_0 and left-most M_0^* are preserved, but the number of Y inserted in the matrix element is reduced by one. Therefore, we can recover the matrix element

$$\langle \varphi, Y(a^1, x_1) Y(a^2, x_2) \ldots Y(a^n, x_n) w \rangle$$

only from the knowledge of the top space M_0.

The proof of Theorem 8.8 goes as follows.

Step 1: We fist construct functionals

$$S \colon W^* \otimes V^{\otimes n} \otimes W \to \mathbb{C}[x_1, \ldots, x_n][x_i^{-1}, (x_i - x_j)^{-1}]$$

that pretend matrix elements by recursion in n.

For $n = 0$, there is a natural pairing

$$S = \langle -, - \rangle : W^* \otimes W \to \mathbb{C}.$$

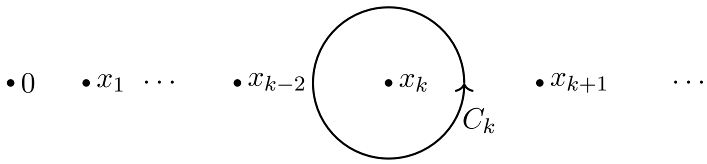
Assuming that S is defined up to $n-1$, we define, for $a^1, \ldots, a^n \in V$, $w \in W$ and $\varphi \in W^*$,

$$\begin{aligned}
&S(\varphi, (a^1, x_1)(a^2, x_2) \cdots (a^n, x_n) w) \\
&= S(o(a^1)^* \varphi, (a^2, x_2) \ldots (a^n, x_n) w) \\
&+ \sum_{k=2}^{n} \sum_{i=0}^{\infty} F_{\deg a^1, i}(x_1, x_k) \cdot S(\varphi, (a^2, x_2) \cdots (a^1_{(i)} a^k, x_k) \cdots (a^n, x_n) w).
\end{aligned}$$

Step 2: It can be shown that the functions S satisfy the following property:

$$\int_{C_{k+1}} S(\varphi, \cdots (a^{k-1}, x_{k-1})(a^k, x_k) \cdots w)(x_{k-1} - x_k)^n dx_{k-1}$$
$$= S(\varphi, \cdots (a^{k-1}_{(n)} a^k, x_k) \cdots w) \tag{8.1}$$

for any k. Here C_k is an integral contour that encloses only x_k, but not others nor 0.



We omitted the numerical factor $\frac{1}{2\pi \mathbb{i}}$ from the integral.
Step 3: We define $\overline{M}$ by the formal span

$$\overline{M} = \mathrm{Span}\left\{ b^1_{(i_1)} \cdots b^l_{(i_l)} w \,\middle|\, b^1, \dots, b^l \in V,\ i_1, \dots, i_l \in \mathbb{Z},\ w \in W \right\}$$

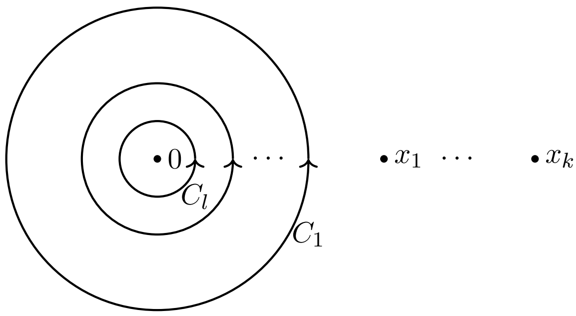
and extend S to $W^* \otimes V^{\otimes n} \otimes \overline{M}$ as follows: for

$$m = b^1_{(i_1)} \cdots b^l_{(i_l)} w,$$

we set

$$S(\varphi, (a^1, x_1) \cdots (a^k, x_k) m)$$
$$= \int_{C_1} \cdots \int_{C_l} S(\varphi, (a^1, x_1) \cdots (a^k, x_k)(b^1, y_1) \cdots (b^l, y_l) w) y_1^{i_i} \cdots y_l^{i_l} dy_1 \cdots dy_l.$$

Here, each y_i is integrated along C_i that encloses 0 and all C_j of $j > i$. The points $x_1, \dots, x_k$ are outside these contours.



It is clear that (8.1 still holds even if we replace $w \in W$ with $m \in \overline{M}$:

$$\int_{C_{k+1}} S(\varphi, \cdots (a^{k-1}, x_{k-1})(a^k, x_k) \cdots m)(x_{k-1} - x_k)^n dx_{k-1}$$
$$= S(\varphi, \cdots (a^{k-1}_{(n)} a^k, x_k) \cdots m). \tag{8.2}$$

Step 4: We define

$$\operatorname{Rad}(\overline{M}) = \left\{ m \in \overline{M} \,\middle|\, S(\varphi, (a^1, x_1) \cdots (a^n, x_n)m) = 0, \ {}^{\text{for all } \varphi \in W^*,}_{a^1, \ldots, a^n \in V, n \in \mathbb{N}} \right\}$$

and

$$M = \overline{M}/\operatorname{Rad}(\overline{M}).$$

It is clear that $\operatorname{Rad}(\overline{M})$ is stable under adding a symbol $a_{(n)}$, $a \in V$, $n \in \mathbb{Z}$ on the left. Therefore, if we define

$$Y_M(a, x) = \sum_{n \in \mathbb{Z}} a_{(n)} x^{-n-1}$$

for $a \in V$, it is considered in $\operatorname{End}(M)[[x^{\pm 1}]]$.

The claim is, of course, that the pair (M, Y_M) is the desired $\mathbb{N}$-gradable V-module. Proving that requires few extra works, but most importantly, (8.2) implements the property

$$Y_M(a_{(n)}b, x) = Y_M(a, x)_{(n)} Y_M(b, x), \quad a, b \in V, \, n \in \mathbb{Z}$$

on M.